ON THE 2D ISENTROPIC EULER SYSTEM WITH UNBOUNDED INITIAL VORTICITY

ZINEB HASSAINIA

Abstract. This paper is devoted to the study of the low Mach number limit for the 2D isentropic Euler system associated to ill-prepared initial data with slow blow up rate on $\log \varepsilon^{-1}$. We prove in particular the strong convergence to the solution of the incompressible Euler system when the vorticity belongs to some weighted BMO spaces allowing unbounded functions. The proof is based on the extension of the result of [6] to a compressible transport model.

Contents

1. Introduction

The equations of motion governing a perfect compressible fluid evolving in the whole space $\mathbb{R}^2$ are given by Euler system:

$$\begin{cases} \rho(\partial_t v + v\cdot\nabla v) + \nabla p = 0,\ t\geq 0,\ x\in\mathbb{R}^2 \\ \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ (v,\rho)_{|t=0} = (v_0,\rho_0). \end{cases}$$

Here, the vector field $v=(v_1,v_2)$ describes the velocity of the fluid particles and the scalar functions p and $\rho>0$ stand for the pressure and the density, respectively. From now onwards, we shall be concerned only with the isentropic case corresponding to the law

$$p=\rho^\gamma,$$

where the parameter $\gamma>1$ is the adiabatic exponent.

Following the idea of Kawashima, Makino and Ukai [24], this system can be symmetrized by using the sound speed c defined by

$$c = 2\frac{\sqrt{\gamma}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}.$$

2000 *Mathematics Subject Classification.* 76N10 ; 35Q35.
Key words and phrases. 2D compressible Euler equations, Incompressible limit, BMO-type spaces.

The main scope of this paper is to deal with the weakly compressible fluid and particularly we intend to get a lower bound for the lifespan and justify the convergence towards the incompressible system. But before reviewing the state of the art and giving a precise statement of our main result we shall briefly describe the way how to get formally the weakly compressible fluid. In broad terms, the basic idea consists in writing the foregoing system around the equilibrium state $(0, c_0)$: let $\varepsilon > 0$ be a small parameter called the Mach number and set

$$v(t,x) = \bar{\gamma} c_0 \varepsilon v_\varepsilon(\varepsilon \bar{\gamma} c_0 t, x) \quad \text{and} \quad c(t,x) = c_0 + \bar{\gamma} c_0 \varepsilon c_\varepsilon(\varepsilon \bar{\gamma} c_0 t, x) \quad \text{with} \quad \bar{\gamma} = \frac{\gamma - 1}{2}.$$

Then the resulting system will be the following

$$\begin{cases} \partial_t v_\varepsilon + v_\varepsilon.\nabla v_\varepsilon + \frac{1}{\varepsilon}\nabla c_\varepsilon + \bar{\gamma} c_\varepsilon \nabla c_\varepsilon = 0, \\ \partial_t c_\varepsilon + v_\varepsilon \cdot \nabla c_\varepsilon + \frac{1}{\varepsilon}\operatorname{div} v_\varepsilon + \bar{\gamma} c_\varepsilon \operatorname{div} v_\varepsilon = 0, \\ (v_\varepsilon, c_\varepsilon)_{|t=0} = (v_{0,\varepsilon}, c_{0,\varepsilon}). \end{cases} \tag{E.C}$$

As we can easily observe this system contains singular terms in ε that might affect dramatically the dynamics when the Mach number is close to zero. For more details about the derivation of the above model we invite the interested reader to consult the papers [16, 18, 9] and the references therein.

From mathematical point of view this system has been intensively investigated in the few last decades. One of the basic problems is the construction of the solutions $(v_\varepsilon, c_\varepsilon)$ in suitable function spaces with a non degenerate time existence and most importantly the asymptotic behavior for small Mach number. Formally, one expects the velocity v_ε to converge to v the solution of the incompressible Euler system given by

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = 0, \\ \operatorname{div} v = 0, \\ v_{|t=0} = v_0. \end{cases} \tag{E.I}$$

As a matter of fact, the singular parts are antisymmetric and do not contribute in the energy estimates built over Sobolev spaces H^s. Accordingly, a uniform time existence can be shown by using just the theory of hyperbolic systems, see [17]. However it is by no means obvious that the constructed solutions will converge to the expected incompressible Euler solution and the problem can be highly non trivial when it is coupled with the geometry of the domain, a fact that we ignore here. In most of the papers dealing with this recurrent subject there are essentially two kinds of hypothesis on the initial data: the first one concerns the well-prepared case where the initial data are assumed to be slightly compressible meaning that $(\operatorname{div} v_{0,\varepsilon}, \nabla c_{0,\varepsilon}) = O(\varepsilon)$ for ε close to zero. In this context it can be proved that the time derivative of the solutions is uniformly bounded and therefore the justification of the incompressible limit follows from Aubin-Lions compactness lemma. For a complete discussion we refer the reader to the papers of Klainerman and Majda [17, 18]. The second class of initial data is the ill-prepared case where the family $(v_{0,\varepsilon}, c_{0,\varepsilon})_\varepsilon$ is assumed to be bounded in Sobolev spaces H^s with $s > 2$ and the incompressible parts of $(v_{0,\varepsilon})_\varepsilon$ converge strongly to some divergence-free vector field v_0 in L^2. In this framework, the main difficulty that one has to face, as regards the incompressible limit, is the propagation of the time derivative $\partial_t v_\varepsilon$ with the speed ε^{-1}, a phenomenon which does not occur in the case of the well-prepared data. To deal with this trouble, Ukai used in [31] the dispersive effects generated by the acoustic waves in order to prove that the compressible part of the velocity and the acoustic term vanish when ε goes to zero. Similar studies but in more complex situations and for various models were accomplished later in different works and for the convenient of the reader we quote here a short list of references [1, 3, 10, 15, 16, 19, 20, 22].

Regarding the lifespan of these solutions, it is well-known that in contrast to the incompressible case where the classical solutions are global in dimension two, the compressible Euler system (E.C) may develop singularities in finite time for some smooth initial data. This was shown in space dimension two by Rammaha [26], and by Sideris [28] for dimension three. It seems that in dimension two we can generically get a lower bound for the lifespan T_ε that goes to infinity for small ε. More precisely, when the initial data are bounded in H^s with $s > 2$ then by taking benefit of the vorticity structure coupled with Strichartz estimates we get,

$$T_\varepsilon \geq C \log\log \varepsilon^{-1}.$$

Besides, we can get precise information on the lifespans when the initial data enjoy some specific structures. In fact, Alinhac [2] showed that in two-dimensional space and for axisymmetric data the lifespan is equivalent to ε^{-1}. Also, for the three-dimensional system, Sideris [29] proved the almost global existence of the solution for potential flows. In other words, it was shown that the lifespan T_ε is bounded below by $\exp(c/\varepsilon)$. To end this short discussion we mention that global existence results were obtained in [12, 27] for some restrictive initial data.

Recently the incompressible limit to (E.C) for ill-prepared initial data lying to the critical Besov space $B^2_{2,1}$ was carried out in [16]. It was also shown that the strong convergence occurs in the space of the initial data. The same program was equally accomplished in dimension three in [15] for the axisymmetric initial data. The fact that the regularity is optimal for the incompressible system will contribute with much more technical difficulties and unfortunately the perturbation theory cannot be easily adapted. In these studies, the geometry of the vorticity is of crucial importance.

In the contributions cited before, the velocity should be in the Lipschitz class uniformly with respect to ε. This constraint was slightly relaxed in [9] by allowing the initial data to be so ill-prepared in order to permit Yudovich solutions for the incompressible system. Recall that these latter solutions are constructed globally in time for (E.I) when the initial vorticity ω_0 belongs to $L^1 \cap L^\infty$, see [33]. In the incompressible framework the vorticity ω, defined for a vector field $v = (v_1, v_2)$ by $\omega = \partial_1 v_2 - \partial_2 v_1$, is advected by the flow,

$$\partial_t \omega + v \cdot \nabla \omega = 0, \quad \Delta v = \nabla^\perp \omega. \tag{1}$$

Working in larger spaces than the Yudovich's one for the system (1) and peculiarly with unbounded vorticity, possibly without uniqueness, is not in general an easy task and often leads to more technical complications. Nevertheless, in the last decade slight progress were done and we shall here comment only some of them which fit with the scope of this paper. For a complete list of references we invite the reader to check the papers [8, 11, 34]. One of the basic result in this subject is due to Vishik in [32] who gave various results when the vorticity belongs to the class B_Γ: a kind of functional space characterized by the slow growth of the partial sum built over the dyadic Fourier blocks. The results of Vishik which cover global and local existence with or without uniqueness depending on some analytic properties of Γ suffer from one inconvenient: the persistence regularity is not proved and an instantaneous loss of regularity may happen.

Recently, Bernicot and Keraani proved in [6] the global existence and uniqueness without any loss of regularity for the incompressible Euler system when the initial vorticity is taken in a weighted BMO space called $LBMO$ and denotes the set of functions with log-*bounded mean oscillations*. This space is strictly larger than L^∞ and smaller than the usual BMO space.

The main task of this paper is to conduct the incompressible limit study for (E.C) when the limiting system (E.I) is posed for initial data lying in the $LBMO$ space. As we shall discuss later we will be also able to generalize the result of [6] for more general spaces. To give a clear statement we need to introduce the $LBMO$ space and a precise discussion will be found in the next section. First, take $f : \mathbb{R}^2 \to \mathbb{R}$ be a locally integrable function. We say that f belongs to BMO space if

$$\|f\|_{BMO} \triangleq \sup_{B\, ball} \fint_B \Big| f - \fint_B f \Big|.$$

Second, we say that f belongs to the space $LBMO$ if

$$\|f\|_{BMO_F} \triangleq \|f\|_{BMO} + \sup_{2B_2 \subset B_1} \frac{|\fint_{B_2} f - \fint_{B_1} f|}{\ln \Big| \dfrac{\ln r_2}{\ln r_1} \Big|} < +\infty,$$

where the supremum is taken over all the pairs of balls $B_2 = B(x_2, r_2)$ and $B_1 = B(x_1, r_1)$ in $\mathbb{R}^2$ with $0 < r_1 \leq \frac{1}{2}$. We have used the notation $\fint_B f$ to refer to the average $\frac{1}{|B|} \int_B f(x)dx$.

Next we shall state our main result in the special case of $LBMO$ space and whose extension will be given in Theorem 4.

Theorem 1. *Let $s,\alpha\in]0,1[$ and $p\in]1,2[$. Consider a family of initial data $(v_{0,\varepsilon},c_{0,\varepsilon})_{0<\varepsilon<1}$ such that there exists a constant $C>0$ which does not depend on ε and verifying*

$$\|(v_{0,\varepsilon},c_{0,\varepsilon})\|_{H^{s+2}}\leq C(\log\varepsilon^{-1})^{\alpha},$$
$$\|\omega_{0,\varepsilon}\|_{L^p\cap LBMO}\leq C.$$

Then, the system (E.C) admits a unique solution $(v_\varepsilon,c_\varepsilon)\in C([0,T_\varepsilon[;H^{s+2})$ with the following properties:

(i) *The lifespan T_ε of the solution satisfies the lower bound:*

$$T_\varepsilon\geqslant\log\log\log\varepsilon^{-1}\triangleq\tilde{T}_\varepsilon,$$

and for all $t\leq\tilde{T}_\varepsilon$ we have

$$\|\omega_\varepsilon(t)\|_{LBMO\cap L^p}\quad\leq\quad C_0e^{C_0t}.\tag{2}$$

Moreover, the compressible and acoustic parts of the solutions converge to zero:

$$\lim_{\varepsilon\to0}\|(\operatorname{div}v_\varepsilon,\nabla c_\varepsilon)\|_{L^1_{\tilde{T}_\varepsilon}L^\infty}=0.$$

(ii) *Assume in addition that $\lim_{\varepsilon\to0}\|\omega_{0,\varepsilon}-\omega_0\|_{L^p}=0$, for some vorticity $\omega_0\in LBMO\cap L^p$. associated to a divergence-free vector field v_0. Then the vortices $(\omega_\varepsilon)_\varepsilon$ converge strongly to the weak solution ω of (1) associated to the initial data ω_0: for all $t\in\mathbb{R}_+$ we have*

$$\lim_{\varepsilon\to0}\|\omega_\varepsilon(t)-\omega(t)\|_{L^q}=0,\quad\forall q\in[p,+\infty[,\tag{3}$$

and

$$\|\omega(t)\|_{LBMO\cap L^p}\quad\leq\quad C_0e^{C_0t}.\tag{4}$$

The constant C_0 depends only on the size of the initial data and does not depend on ε.

Before giving a brief account of the proof, we shall summarize some comments in order to clarify some points in the theorem.

Remark 1. (i) *Theorem 1 recovers the result stated in [6] for $LBMO\cap L^p$ space according to the estimate (4).*

(ii) *The estimate (3) can be translated to the velocity according to the Biot-Savart law (7) as follows*

$$\lim_{\varepsilon\to0}\|\mathbb{P}v_\varepsilon-v\|_{L^\infty_tW^{1,r}\cap L^\infty}=0\quad\forall r\in\Big[\frac{2p}{2-p},+\infty\Big[.$$

where $\mathbb{P}v_\varepsilon=v_\varepsilon-\nabla\Delta^{-1}\operatorname{div}v_\varepsilon$ denotes the Leray's projector over solenoidal vector fields.

(iii) *We can generically construct a family $(v_{0,\varepsilon})$ satisfying the assumptions of Theorem 1. In fact, let v_0 be a divergence-free vector field with $\omega_0=\operatorname{curl}v_0\in L^p\cap LBMO$. Take two functions $\chi,\rho\in C_0^\infty(\mathbb{R}^2)$ with*

$$\rho\geq0\quad\textit{and}\quad\int_{\mathbb{R}^2}\rho(x)dx=1.$$

Denote by $(\rho_k)_{k\in\mathbb{N}^*}$ the usual mollifiers:

$$\rho_k(x)\triangleq k^2\rho(kx).$$

For $R>0$, we set

$$v_{0,\varepsilon}=\rho_k*\Big(\chi(\frac{\cdot}{R})v_0\Big).$$

By the convolution laws we obtain

$$\|v_{0,\varepsilon}\|_{H^{s+2}}\quad\leq\quad Ck^{s+2}R\|v_0\|_{L^\infty}.$$

We choose carefully k and R with slight growth with respect to ε in order to get

$$k^{s+2}R\leq C_0(\ln\varepsilon^{-1})^{\alpha}.$$

The uniform boundedness of $(\omega_{0,\varepsilon})$ in the space $LBMO$ is more subtle and will be the object of Proposition 1 and Proposition 2 in the next section.

Let us now outline the basic ideas for the proof of Theorem 1. It is founded on two main ingredients: the first one which is the most relevant and has an interest in itself concerns the persistence of the regularity $LBMO$ for a compressible transport model governing the vorticity,

$$\partial_t \omega_\varepsilon + v_\varepsilon \cdot \nabla \omega_\varepsilon + \omega_\varepsilon \mathrm{div}\, v_\varepsilon = 0.$$

In the incompressible case (1), Bernicot and Keraani have shown recently in [6] the following estimate

$$\|\omega(t)\|_{LBMO\cap L^p} \quad \leq \quad C\|\omega_0\|_{LBMO\cap L^p}\Big(1+\int_0^t \|v(\tau)\|_{LL} d\tau\Big). \tag{5}$$

Where LL refers to the norm associated to the log-Lipshitz space.
Our goal consists in extending this result to the compressible model cited before. To do so, we shall proceed in the spirit of the work [6] by following the dynamics of the oscillations and especially understand the interaction between them and how the global mass is distributed. However, the lack of the incompressibility of the velocity and the quadratic structure of the nonlinearity $\omega_\varepsilon \mathrm{div}\, v_\varepsilon$ will bring more technical difficulties that we should carefully analyze. Our result whose extension will be given later in Theorem 2 reads as follows,

$$\begin{aligned}\|\omega_\varepsilon(t)\|_{LBMO\cap L^p} \quad &\leq \quad C\|\omega_{0,\varepsilon}\|_{LBMO\cap L^p}\Big(1+\int_0^t \|v_\varepsilon(\tau)\|_{LL} d\tau\Big)\\ &\times \quad \Big(1+\|\mathrm{div}\, v_\varepsilon\|_{L^1_t C^s}\int_0^t \|v_\varepsilon(\tau)\|_{LL} d\tau\Big)e^{C\|\mathrm{div}\, v_\varepsilon\|_{L^1_t L^\infty}}.\end{aligned}$$

From this estimate the result (5) follows easily by taking $\mathrm{div}\, v_\varepsilon = 0$. Now to prove this result we shall first filtrate the compressible part and then reduce the problem to the establishment of a logarithmic estimate for the composition in the space $LBMO$ but with a flow which does not necessarily preserve the Lebesgue measure. For this part we follow the ideas of [6]. Once this logarithmic estimate is proven we should come back to the real solution and thus we are led to establish some law products invoking some weighted LMO spaces acting as multipliers of $LBMO$ space.

The second ingredient of the proof of Theorem 1 is the use of the Strichartz estimates which are an efficient tool to deal with the so ill-prepared initial data. As it has already been mentioned, this fact was used in [9] for Yudovich solutions and here we follow the same strategy but with slight modifications for the strong convergence. This is done directly by manipulating the vorticity equation.

The remainder of this paper is organized as follows. In the next section we recall basic results about Littlewood-Paley operators, Besov spaces and gather some preliminary estimates. We shall also introduce some functional spaces and prove some of their basic properties. In Section 3 we shall examine the regularity of the flow map and establish a logarithmic estimate for the compressible transport model. Section 4 is devoted to some classical energy estimate for the system (E.C) and the corresponding Strichartz estimates. In the last section, we generalize the result of Theorem 1 and give the proofs. We close this paper with an appendix covering the proof of some technical lemmas.

2. Functional tool box

In this section, we shall recall the definition of the frequency localization operators, some of their elementary properties and the Besov spaces. We will also introduce some function spaces and discuss few basic results that will be used later.
First of all, we fix some notations that will be intensively used in this paper.

- In what follows, C stands for some real positive constant which may be different in each occurrence and C_0 a constant which depends on the initial data.
- For any X and Y, the notation $X \lesssim Y$ means that there exists a positive universal constant C such that $X \leq CY$.

- For a ball B and $\lambda > 0$, λB denotes the ball that is concentric with B and whose radius is λ times the radius of B.
- We will denote the mean value of f over the ball B by

$$\fint_B f \triangleq \frac{1}{|B|}\int_B f(x)\,dx.$$

- For $p \in [1,\infty]$, the notation $L^p_T X$ stands for the set of measurable functions $f : [0,T] \to X$ such that $t \longmapsto \|f(t)\|_X$ belongs to $L^p([0,T])$.

2.1. **Littelwood-Paley theory.** Let us recall briefly the classical dyadic partition of the unity, for a proof see for instance [7] : there exists two positive radial functions $\chi \in \mathcal{D}(\mathbb{R}^2)$ and $\varphi \in \mathcal{D}(\mathbb{R}^2\backslash\{0\})$ such that

$$\forall \xi \in \mathbb{R}^2, \quad \chi(\xi) + \sum_{q\geq 0}\varphi(2^{-q}\xi) = 1;$$
$$\forall \xi \in \mathbb{R}^2\backslash\{0\}, \quad \sum_{q\in\mathbf{Z}}\varphi(2^{-q}\xi) = 1;$$
$$|j-q| \geq 2 \Rightarrow \quad \mathrm{Supp}\ \varphi(2^{-j}\cdot) \cap \mathrm{Supp}\ \varphi(2^{-q}\cdot) = \varnothing;$$
$$q \geq 1 \Rightarrow \quad \mathrm{Supp}\ \chi \cap \mathrm{Supp}\ \varphi(2^{-q}\cdot) = \varnothing.$$

For every $u \in \mathcal{S}'(\mathbb{R}^2)$ one defines the non homogeneous Littlewood-Paley operators by,

$$\Delta_{-1}v = \mathcal{F}^{-1}\big(\chi\hat{v}\big), \quad \forall q \in \mathbb{N} \quad \Delta_q v = \mathcal{F}^{-1}\big(\varphi(2^{-q}\cdot)\hat{v}\big) \quad \text{and} \quad S_q v = \sum_{-1\leq j\leq q-1}\Delta_j v.$$

Symilarly, we define the homogeneous operators by

$$\forall q \in \mathbb{Z} \quad \dot{\Delta}_q v = \mathcal{F}^{-1}\big(\varphi(2^{-q}\cdot)\hat{v}\big) \quad \text{and} \quad \dot{S}_q v = \sum_{-\infty\leq j\leq q-1}\dot{\Delta}_j v.$$

We notice that these operators map continuously L^p to itself uniformly with respect to q and p. Furthermore, one can easily check that for every tempered distribution v, we have

$$v = \sum_{q\geq -1}\Delta_q v,$$

and for all $v \in \mathcal{S}'(\mathbb{R}^2)/\{\mathcal{P}[\mathbb{R}^2]\}$

$$v = \sum_{q\in\mathbb{Z}}\dot{\Delta}_q v,$$

where $\mathcal{P}[\mathbb{R}^2]$ is the space of polynomials.
The following lemma (referred in what follows as Bernstein inequalities) describes how the derivatives act on spectrally localized functions.

Lemma 1. *There exists a constant $C > 0$ such that for all $q \in N, k \in \mathbb{N}, 1 \leq a \leq b \leq \infty$ and for every tempered distribution u we have*

$$\sup_{|\alpha|\leq k}\|\partial^\alpha S_q u\|_{L^b} \leq C^k 2^{q(k+2(\frac{1}{a}-\frac{1}{b}))}\|S_q u\|_{L^a},$$
$$C^{-k}2^{qk}\|\dot{\Delta}_q u\|_{L^b} \leq \sup_{|\alpha|=k}\|\partial^\alpha\dot{\Delta}_q u\|_{L^b} \leq C^k 2^{qk}\|\dot{\Delta}_q u\|_{L^b}.$$

Based on Littlewood-Paley operators, we can define Besov spaces as follows. Let $(p,r) \in [1,+\infty]^2$ and $s \in \mathbb{R}$. The non homogeneous Besov space $B^s_{p,r}$ is the set of tempered distributions v such that

$$\|v\|_{B^s_{p,r}} \triangleq \Big\|\big(2^{qs}\|\Delta_q v\|_{L^p}\big)_{q\in\mathbb{Z}}\Big\|_{\ell^r(\mathbb{Z})} < +\infty.$$

The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as the set of $\mathcal{S}'(\mathbb{R}^2)/\{\mathcal{P}[\mathbb{R}^2]\}$ such that

$$\|v\|_{\dot{B}^s_{p,r}} \triangleq \Big\|\big(2^{qs}\|\dot{\Delta}_q v\|_{L^p}\big)_{q\in\mathbb{Z}}\Big\|_{\ell^r(\mathbb{Z})} < +\infty.$$

We point out that, for a strictly positive non integer real number s the Besov space $B^s_{\infty,\infty}$ coincides with the usual Hölder space C^s. For $s\in]0,1[$, this means that

$$\|v\|_{B^s_{\infty,\infty}} \lesssim \|v\|_{L^\infty} + \sup_{x\neq y}\frac{|v(x)-v(y)|}{|x-y|^s} \lesssim \|v\|_{B^s_{\infty,\infty}}.$$

Also we can identify $B^s_{2,2}$ with the Sobolev space H^s for all $s\in\mathbb{R}$.
The following embeddings are an easy consequence of Bernstein inequalities,

$$B^s_{p_1,r_1}\hookrightarrow B^{s+2(\frac{1}{p_2}-\frac{1}{p_1})}_{p_2,r_2}\quad p_1\le p_2\quad\text{and}\quad r_1\le r_2.$$

Next, we recall the log-Lipschitz space, denoted by LL. It is the set of bounded functions v such that

$$\|v\|_{LL}\triangleq \sup_{0<|x-y|<1}\frac{|v(x)-v(y)|}{|x-y|\log\frac{e}{|x-y|}}<+\infty.$$

Note that the space $B^1_{\infty,\infty}$ is a subspace of LL. More precisely, we have the following inequality, see [4] for instance.

$$\|v\|_{LL}\lesssim \|\nabla v\|_{B^0_{\infty,\infty}}. \tag{6}$$

If in addition v is divergence-free and under sufficient conditions of integrability, the velocity v is determined by the vorticity $\omega\triangleq \mathrm{rot}v$ by means of the Biot-Savart law

$$v(x)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\omega(y)dy. \tag{7}$$

The following result is a deep estimate of harmonic analysis and related to the singular integrals of Calderón-Zygmund type,

$$\|\nabla v\|_{L^p}\le C\frac{p^2}{p-1}\|\omega\|_{L^p}. \tag{8}$$

where C is a universal constant and $p\in]1,\infty[$.

Lemma 2. *Let v be a smooth vector field and ω its vorticity. Define the compressible part of v by*

$$\mathbb{Q}v\triangleq \nabla\Delta^{-1}\mathrm{div}\, v.$$

Then,

$$\|v\|_{LL}\lesssim \|\mathbb{Q}v\|_{L^\infty}+\|\omega\|_{L^p\cap B^0_{\infty,\infty}}+\|\mathrm{div}\, v\|_{B^0_{\infty,\infty}}. \tag{9}$$

Proof. The identity $v=\mathbb{P}v+\mathbb{Q}v$ and the estimate (6) ensure that

$$\|v\|_{LL}\ \lesssim\ \|\nabla\mathbb{P}v\|_{B^0_{\infty,\infty}}+\|\nabla\mathbb{Q}v\|_{B^0_{\infty,\infty}}.$$

Bernstein inequality, the continuity of $\dot\Delta_q\mathbb{P}:L^p\longrightarrow L^p$, $\forall p\in[1,\infty]$ uniformly in q and the classical fact $\|\dot\Delta_q v\|_{L^\infty}\sim 2^{-q}\|\dot\Delta_q\omega\|_{L^\infty}$ give

$$\begin{aligned}\|\nabla\mathbb{P}v\|_{B^0_{\infty,\infty}} &\le \|\Delta_{-1}\nabla\mathbb{P}v\|_{L^\infty}+\sup_{q\in\mathbb{N}}\|\dot\Delta_q\nabla\mathbb{P}v\|_{L^\infty}\\ &\lesssim \|\Delta_{-1}\nabla\mathbb{P}v\|_{L^p}+\sup_{q\in\mathbb{N}}2^q\|\dot\Delta_q\mathbb{P}v\|_{L^\infty}\\ &\lesssim \|\nabla\mathbb{P}v\|_{L^p}+\sup_{q\in\mathbb{N}}\|\dot\Delta_q\omega\|_{L^\infty}.\end{aligned}$$

Since the incompressible part $\mathbb{P}v$ has the same vorticity ω of the total velocity:

$$\mathrm{curl}\ \mathbb{P}v=\omega,$$

then by the inequality (8) we get

$$\|\nabla\mathbb{P}v\|_{B^0_{\infty,\infty}}\lesssim \|\omega\|_{L^p}+\|\omega\|_{B^0_{\infty,\infty}}.$$

On other hand,

$$\begin{aligned}\|\nabla\mathbb{Q}v\|_{B^0_{\infty,\infty}} &\leq \|\Delta_{-1}\nabla\mathbb{Q}v\|_{L^\infty}+\sup_{q\in\mathbb{N}}\|\dot\Delta_q\nabla^2\Delta^{-1}\mathrm{div}\, v\|_{L^\infty}\\ &\leq \|\mathbb{Q}v\|_{L^\infty}+\|\mathrm{div}\, v\|_{B^0_{\infty,\infty}}.\end{aligned}$$

This concludes the proof of the lemma. □

The following result generalizes the classical Gronwall inequality, it will be very useful in the proof of the lifespan part of Theorem 3. For its proof see Lemma 5.2.1 in [7] .

Lemma 3. *[Osgood Lemma] Let $a, C > 0$, $\gamma : [t_0, T] \to \mathbb{R}_+$ be a locally integrable function and $\mu : [a, +\infty[\longrightarrow \mathbb{R}_+$ be a continuous non-decreasing function. Let $\rho : [t_0, T] \to [a, +\infty[$ be a measurable, positive function satisfying*

$$\rho(t) \leq C + \int_{t_0}^t \gamma(s)\mu(\rho(s))ds.$$

Set $\mathcal{M}(y) = \displaystyle\int_a^y \frac{dx}{\mu(x)}$ and assume that $\lim\limits_{y\to+\infty}\mathcal{M}(y) = +\infty$. Then

$$\forall t \in [t_0, T], \quad \rho(t) \leq \mathcal{M}^{-1}\Big(\mathcal{M}(C) + \int_{t_0}^t \gamma(s)ds\Big).$$

Next, we will introduce a new space which play a crucial role in the study of Euler equations as we will see later.

2.2. **The BMO_F space.** The main goal of this section is to introduce the weighted BMO spaces denoted by BMO_F where the function F measures the rate between two oscillations. Thereafter, we shall focus on the analysis of some of their useful topological properties.

To start with, let us recall the classical BMO space (bounded mean oscillation), which is nothing but the set of locally integrable function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\|f\|_{BMO} \triangleq \sup_B \fint_B \Big|f - \fint_B f\Big| < +\infty,$$

where B runs over all the balls in $\mathbb{R}^2$.
It is well-known that the quotient space BMO space by the constants is a Banach space. Moreover, the BMO space enjoys with the following classical properties:

(i) BMO is imbricated between the space of bounded functions and the $B^0_{\infty,\infty}$ space, that is,

$$L^\infty \hookrightarrow BMO \hookrightarrow B^0_{\infty,\infty}.$$

(ii) The unit ball of the BMO space is a weakly compact set.

Another interesting property of the BMO space concerns the rate between two oscillations which is estimated as follows: take two balls $B_1 = B(x_1, r_1)$ and $B_2 = B(x_2, r_2)$ such that $2B_2 \subset B_1$, then

$$\Big|\fint_{B_2} f - \fint_{B_1} f\Big| \lesssim \ln(1 + \frac{r_1}{r_2})\|f\|_{BMO}. \tag{10}$$

It is worthy pointing out that the local well-posedness theory for the incompressible Euler equations is still an open problem when the initial vorticity belongs to the BMO space. However, it was proved recently in [6] that the global well-posedness can be achieved for the $LBMO$ space, which is larger than the bounded functions and smaller than the BMO space. To be precise this space is defined by

$$\|f\|_{LBMO} \triangleq \|f\|_{BMO} + \sup_{2B_2\subset B_1} \frac{|\fint_{B_2} f - \fint_{B_1} f|}{\ln\Big|\dfrac{\ln r_2}{\ln r_1}\Big|} < +\infty,$$

where the radius r_1 of B_1 is smaller than $\frac{1}{2}$. It seems that we can perform the same result for more general spaces by replacing the "outside" logarithm in the second part of the norm by a general function F which must satisfy some special assumptions listed below. We mention that a similar extension was done in the paper [5]. Before stating the definition of these spaces we need the following notions.

Definition 1. *Let $F : [1, +\infty[\longrightarrow [1, +\infty[$ be a nondecreasing function.*

- *We say that F belongs to the class $\mathcal{F}$ if there exits a constant $C > 0$ such that the following properties hold true:*
  (a) *Blow up at infinity:* $\lim_{x\to 0} F(x) = +\infty$.
  (b) *Asymptotic behavior: For any $\lambda \in [1, +\infty[$ and $x \in [\lambda, +\infty[$, we have*
  $$\int_x^{+\infty} e^{-\frac{y}{\lambda}} F(y)\,dy \le C\,\lambda\, e^{-\frac{x}{\lambda}} F(x).$$
  (c) *Polynomial growth: For all $(x, y) \in ([1, +\infty[)^2$*
  $$F(xy) \le CF(x)F(y).$$
- *We say that F belongs to the class $\mathcal{F}'$ if it belongs to $\mathcal{F}$ and satisfies the Osgood condition:*
  $$\int_1^{\infty} \frac{dx}{xF(x)} = +\infty.$$

Now we shall give some elementary facts listed in the following remark.

Remark 2. (i) *Note that the condition (c) in the previous definition implies that the function F has at most a polynomial growth. More precisely, there exists $\alpha > 0$ such that*
$$F(x) \le Cx^{\alpha} \quad \forall x \in [1, +\infty[. \tag{11}$$

(ii) *It turns out from the monotony of F that*
$$F(n)e^{-\frac{n+1}{\lambda}} \le \int_n^{n+1} F(x)e^{-\frac{x}{\lambda}} dx \le F(n+1)e^{-\frac{n}{\lambda}}.$$
Thus, we get the equivalence
$$e^{-\frac{1}{\lambda}} \sum_{n\ge N} e^{-\frac{n}{\lambda}} F(n) \le \int_N^{+\infty} e^{-\frac{x}{\lambda}} F(x)dx \le e^{\frac{1}{\lambda}} \sum_{n\ge N} e^{-\frac{n}{\lambda}} F(n).$$
Consequently, using the asymptotic behavior of F described by the point (b) and making a change of variable we obtain for all $a, b \in \mathbb{R}_+^*$,
$$\frac{1}{C} \sum_{n>N} e^{-\frac{n}{\lambda}} F\Big(\frac{n+a}{b}\Big) \le \int_N^{+\infty} e^{-\frac{x}{\lambda}} F(\frac{x+a}{b})dx \le C\lambda e^{-\frac{N}{\lambda}} F\Big(\frac{N+a}{b}\Big). \tag{12}$$

(iii) *Several examples of functions belonging to the class $\mathcal{F}'$ can be found, for instance, we mention: $x \longmapsto 1 + \ln^{\alpha}(x)$ with $0 < \alpha \le 1$; $x \longmapsto 1 + \ln\ln(e + x)\ln(x)$.*

(iv) *$\mathcal{F}$ is strictly embedded in $\mathcal{F}'$: For any $\beta > 0$, the function $x \longmapsto x^{\beta}$ belongs to the class $\mathcal{F}\backslash\mathcal{F}'$.*

The BMO_F space is given by the following definition.

Definition 2. *Let F in $\mathcal{F}$, we denote by BMO_F the space of the locally integrable functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that*
$$\|f\|_{BMO_F} = \|f\|_{BMO} + \sup_{B_1,B_2} \frac{|\fint_{B_2} f - \fint_{B_1} f|}{F\Big(\dfrac{1-\ln r_2}{1-\ln r_1}\Big)} < +\infty,$$
where the supremum is taken over all the pairs of balls $B_2 = B(x_2, r_2)$ and $B_1 = B(x_1, r_1)$ in $\mathbb{R}^2$ with $0 < r_1 \le 1$ and $2B_2 \subset B_1$.

In the next proposition we shall deal with some topological properties for the BMO_F spaces.

Proposition 1. *The following properties hold true.*

(i) *The space BMO_F is complete, included in BMO and containing $L^{\infty}(\mathbb{R}^2)$.*

(ii) *If* $F, G \in \mathcal{F}$ *with* $F \lesssim G$ *then* $BMO_F \hookrightarrow BMO_G$.
(iii) *If* $\ln(1+x) \lesssim F(x), \forall x \geq 1,$ *then* L^∞ *is strictly embedded in* BMO_F.
(iv) *Let* $p \in]1, \infty]$, *then* $BMO_F \cap L^p$ *is weakly compact.*
(v) *For* $g \in L^1$ *and* $f \in BMO_F$ *one has*

$$\|g * f\|_{BMO_F} \leq \|g\|_{L^1} \|f\|_{BMO_F}.$$

Proof. (i) The two embeddings are straightforward. For the completeness of the space we consider a Cauchy sequence $(u_n)_n$ in BMO_F. Since BMO_F is contained in BMO which is complete, then this sequence converges in BMO and then in L^1_{loc} (see [14] for instance). Using the definition of the second term of the BMO_F norm and the convergence in L^1_{loc}, we get the convergence in BMO_F.
(ii) The embedding is obvious from the definition of the second term of the BMO_F norm.
(iii) According to the assertion (ii) we have the embedding $LBMO = BMO_{1+\ln} \hookrightarrow BMO_F$. Now, we conclude by the result of [6] where it is proved that the unbounded function defined by

$$x \longmapsto \begin{cases} \ln(1 + \ln|x|) & \text{if} \quad |x| \leq 1 \\ 0 \quad \text{if} \quad |x| \geq 1. \end{cases}$$

belongs to $BMO_{1+\ln}$.

(iv) Let (w_n) be a bounded sequence of $BMO_F \cap L^p$, that is

$$\sup_n \|w_n\|_{BMO_F \cap L^p} = M < \infty.$$

We shall prove that up to an extraction we can find a subsequence denoted also by (w_n) which converges weakly to some $w \in BMO_F \cap L^p$. The bound of $(w_n)_n$ in L^p implies the existence of a subsequence, denoted also by $(w_n)_n$, and a function $w \in L^p$ such that (w_n) converges weakly in L^p and consequently for all $B = B(x, r)$ we have

$$\lim_{n \to +\infty} \fint_B w_n dx = \fint_B w dx. \tag{13}$$

In addition we have

$$\|w\|_{L^p} \leq \lim_{n\to\infty} \inf \|w_n\|_{L^p} \leq M.$$

Let B_1 and B_2 be two balls in $\mathbb{R}^2$ such that $2B_2 \subset B_1$ and $0 < r_1 \leq 1$. As F is larger than 1 we may write

$$\begin{aligned} \frac{\left| \fint_{B_1} w - \fint_{B_2} w \right|}{F\left(\frac{1-\ln r_2}{1-\ln r_1}\right)} &\leq \left| \fint_{B_1} (w - w_n) - \fint_{B_2} (w - w_n) \right| + \frac{\left| \fint_{B_1} w_n - \fint_{B_2} w_n \right|}{F\left(\frac{1-\ln r_2}{1-\ln r_1}\right)} \\ &\leq \left| \fint_{B_1} (w - w_n) \right| + \left| \fint_{B_2} (w - w_n) \right| + M. \end{aligned}$$

Hence, we get from (13) that

$$\frac{\left| \fint_{B_1} w - \fint_{B_2} w \right|}{F\left(\frac{1-\ln r_2}{1-\ln r_1}\right)} \leq M.$$

Moreover, from the weak compactness in the BMO space we have

$$\|w\|_{BMO} \leq \lim_{n\to\infty} \inf \|w_n\|_{BMO} \leq M.$$

(v) The result follows immediately from the identity

$$x \longmapsto \fint_{B(x,r)} (g * f) = \Big(g * \fint_{B(.,r)} f\Big)(x) \quad \forall r > 0.$$

$\square$

Remark 3. *By using the Hölder inequality we observe that for any ball B of radius r we have*
$$\fint_B |f - \fint_B f| \le Cr^{-\frac{2}{p}} \|f\|_{L^p}.$$
In this respect we will only need to deal with balls whose radius is smaller than a universal constant, say 1.

The following proposition gives a rigorous justification for the choice of the initial data given by Remark 1.

Proposition 2. *Let $1 < p < 2$, $R > 1$ and χ a be smooth compactly supported function equal to 1 in the neighborhood of the unit ball. Let $\omega \in BMO_F \cap L^p$ be the vorticity of a divergence-free vector filed v. Then*
$$\Big\|\mathrm{rot}\Big(\chi(\frac{\cdot}{R})v\Big)\Big\|_{BMO_F \cap L^p} \lesssim \big(\|\chi\|_{L^\infty} + \|\nabla\chi\|_{L^\infty}\big)\|\omega\|_{BMO_F\cap L^p},$$
and
$$\lim_{R\to+\infty} \Big\|\mathrm{rot}\Big(\chi(\frac{\cdot}{R})v\Big) - \omega\Big\|_{L^p} = 0.$$

Proof. It is obvious that the term $\mathrm{rot}\big(\chi(\frac{\cdot}{R})v\big)$ can be splitted into
$$\mathrm{rot}\Big(\chi(\frac{x}{R})v\Big)(x) = \chi(\frac{x}{R})\omega(x) + \frac{1}{R}\nabla^\perp\chi(\frac{x}{R})v(x). \tag{14}$$
By Hölder inequality, it follows that
$$\Big\|\mathrm{rot}\Big(\chi(\frac{\cdot}{R})v\Big)\Big\|_{L^p} \le \|\chi\|_{L^\infty}\|\omega\|_{L^p} + \frac{1}{R}\big\|\nabla\chi(\frac{\cdot}{R})\big\|_{L^2}\|v\|_{L^{\frac{2p}{2-p}}}. \tag{15}$$
In view of the Biot-Savart law and the classical Hardy-Littlewood- Sobolev inequality we have
$$\|v\|_{L^{\frac{2p}{2-p}}} \lesssim \|\omega\|_{L^p}.$$
Inserting this in (15) we conclude that
$$\Big\|\mathrm{rot}\Big(\chi(\frac{\cdot}{R})v\Big)\Big\|_{L^p} \le \big(\|\chi\|_{L^\infty} + \|\nabla\chi\|_{L^\infty}\big)\|\omega\|_{L^p}.$$
For the BMO part of the norm we use (14) combined with the embedding $L^\infty \hookrightarrow BMO$,
$$\begin{aligned}\Big\|\mathrm{rot}\Big(\chi(\frac{\cdot}{R})v\Big)\Big\|_{BMO} &\le \|\chi(\frac{\cdot}{R})\omega\|_{BMO} + \frac{1}{R}\|\nabla^\perp\chi(\frac{\cdot}{R})\cdot v\|_{BMO}\\ &\lesssim \|\chi(\frac{\cdot}{R})\omega\|_{BMO} + \frac{1}{R}\|\nabla\chi\|_{L^\infty}\|v\|_{L^\infty}.\end{aligned} \tag{16}$$
As $1 < p < 2$, the Biot-Savart law ensures that
$$\|v\|_{L^\infty} \lesssim \|\omega\|_{L^p\cap L^{2p}}.$$
Recall the classical result of interpolation, see [14] for instance:
$$\|\omega\|_{L^q} \lesssim \|\omega\|_{L^p}^{\frac{p}{q}}\|\omega\|_{BMO}^{1-\frac{p}{q}} \quad \forall q \in [p, +\infty[. \tag{17}$$
Combining the preceding two inequalities we get
$$\begin{aligned}\|v\|_{L^\infty} &\lesssim \|\omega\|_{L^p}^{\frac{1}{2}}\|\omega\|_{BMO}^{\frac{1}{2}} + \|\omega\|_{L^p}\\ &\lesssim \|\omega\|_{BMO\cap L^p}.\end{aligned} \tag{18}$$
To estimate the first term of the right-hand side term of inequality (16) we write
$$\begin{aligned}\chi(\frac{x}{R})\omega(x) - \fint_B \chi(\frac{y}{R})\omega(y)dy &= \omega(x)\Big(\chi(\frac{x}{R}) - \fint_B \chi(\frac{y}{R})dy\Big)\\ &+ \Big(\fint_B \chi(\frac{y}{R})dy\Big)\Big(\omega(x) - \fint_B \omega(y)dy\Big)\\ &+ \fint_B \omega(y)\Big(\Big(\fint_B \chi(\frac{z}{R})dz\Big) - \chi(\frac{y}{R})\Big)dy.\end{aligned}$$

Hence,

$$\begin{aligned}
\fint_B \Big|\chi(\frac{x}{R})\omega(x)dx - \fint_B \chi(\frac{y}{R})\omega(y)dy\Big| &\le 2\fint_B |\omega(x)|\Big|\chi(\frac{x}{R}) - \fint_B \chi(\frac{y}{R})dy\Big|dx \\
&+ \Big|\fint_B \chi(\frac{x}{R})dx\Big|\fint_B \Big|\omega - \fint_B \omega\Big| \\
&\lesssim \fint_B\fint_B |\omega(x)|\Big|\chi(\frac{x}{R}) - \chi(\frac{y}{R})\Big|dydx + \|\chi\|_{L^\infty}\|\omega\|_{BMO}.
\end{aligned}$$

By the mean value theorem, we get

$$\begin{aligned}
\fint_B\fint_B |\omega(x)|\Big|\chi(\frac{x}{R}) - \chi(\frac{y}{R})\Big|dydx &\le \|\nabla\chi\|_{L^\infty}\fint_B\fint_B |\omega(x)|\Big|\frac{x-y}{R}\Big|dydx \\
&\lesssim r\|\nabla\chi\|_{L^\infty}\fint_B |\omega(x)|dx.
\end{aligned}$$

Using Hölder inequality, the facts $p>1$, $r<1$, and inequality (17) we find

$$\begin{aligned}
\fint_B\fint_B |\omega(x)|\Big|\chi(\frac{x}{R}) - \chi(\frac{y}{R})\Big|dydx &\lesssim r^{1-1/p}\|\nabla\chi\|_{L^\infty}\|\omega\|_{L^{2p}} \\
&\lesssim \|\nabla\chi\|_{L^\infty}\|\omega\|_{BMO\cap L^p}.
\end{aligned}$$

Concerning the the second term of the norm in BMO_F we start with the identity,

$$\begin{aligned}
\fint_{B_2}\chi(\frac{x}{R})\omega(x)dx - \fint_{B_1}\chi(\frac{y}{R})\omega(y)dy &= \fint_{B_2}\chi(\frac{x}{R})\Big(\omega(x) - \fint_{B_2}\omega(y)dy\Big) \\
&+ \Big(\fint_{B_2}\chi(\frac{x}{R})dx\Big)\Big(\fint_{B_2}\omega(y)dy - \fint_{B_1}\omega(y)dy\Big) \\
&+ \Big(\fint_{B_1}\omega(y)dy\Big)\Big(\fint_{B_2}\chi(\frac{x}{R})dx - \fint_{B_1}\chi(\frac{x}{R})dx\Big) \\
&+ \fint_{B_1}\chi(\frac{x}{R})\Big(\Big(\fint_{B_1}\omega(y)dy\Big) - \omega(x)\Big)dx \\
&\triangleq \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4.
\end{aligned}$$

Using the definition of BMO_F space we get

$$\frac{|\mathrm{I}_1| + |\mathrm{I}_2| + |\mathrm{I}_4|}{F\Big(\dfrac{1-\ln r_2}{1-\ln r_1}\Big)} \le \|\chi\|_{L^\infty}\|\omega\|_{BMO_F}.$$

For the remainder term I_3 we proceed as follows: we take $x_0 \in B_1 \cap B_2$ and we use the main value Theorem with $F \ge 1$

$$\begin{aligned}
\frac{|\mathrm{I}_3|}{F\Big(\dfrac{1-\ln r_2}{1-\ln r_1}\Big)} &\lesssim r_1^{-\frac{1}{p}}\|\omega\|_{L^{2p}}\Big(\fint_{B_2}\Big|\chi(\frac{x}{R}) - \chi(\frac{x_0}{R})\Big|dx + \fint_{B_1}\Big|\chi(\frac{x}{R}) - \chi(\frac{x_0}{R})\Big|dx\Big) \\
&\lesssim r_1^{-\frac{1}{p}}\|\omega\|_{L^{2p}}\|\nabla\chi\|_{L^\infty}\big(r_2/R + r_1/R\big) \\
&\lesssim \|\omega\|_{BMO\cap L^p}\|\nabla\chi\|_{L^\infty}.
\end{aligned}$$

We have now to check the strong convergence in L^p space. By considering the identity (14), Hölder inequality and the fact that $\chi \equiv 1$ in the neighborhood of the unit ball,

$$\begin{aligned}
\Big\|\mathrm{rot}\Big(\chi(\frac{\cdot}{R})v\Big) - \omega\Big\|_{L^p} &\le \big\|\chi(\frac{\cdot}{R}) - 1\big\|_{L^\infty}\|\omega\|_{L^p(B^c(0,R))} + \frac{1}{R}\|\nabla^\perp\chi(\frac{\cdot}{R})\|_{L^2}\|v\|_{L^{\frac{2p}{2-p}}(B^c(0,R))} \\
&\lesssim \|\omega\|_{L^p(B^c(0,R))} + \|\nabla\chi\|_{L^\infty}\|v\|_{L^{\frac{2p}{2-p}}(B^c(0,R))}.
\end{aligned}$$

Passing to the limit completes the proof of the desired result. ∎

2.3. LMO_F **spaces and law products.** Here we endeavor to define a functional space whose elements can be served as pointwise multipliers for BMO_F space. In this context, we point out that BMO is stable by multiplication by $LMO \cap L^\infty$ functions, we refer to [25] for the proof. Where LMO is a subspace of BMO, not comparable to L^∞ and equipped with the semi-norm

$$\|f\|_{LMO} \triangleq \sup_B |\ln r| \fint_B \Big|f - \fint_B f\Big|.$$

In this definition B runs over the balls of radius lesser than 1. In order to validate a similar result for the BMO_F space, we have to define the following function space.

Definition 3. *Let F be in the class $\mathcal{F}$ and $f \in L^1_{loc}(\mathbb{R}^2, \mathbb{R})$, we say that f belongs to the LMO_F space if*

$$\|f\|_{LMO_F} = \sup_{\substack{B\\ r\leq 1}} F(1-\ln r) \fint_B \Big|f - \fint_B f\Big| + \sup_{\substack{2B_2\subset B_1\\ r_1\leq 1}} F(1-\ln r_1)\Big|\fint_{B_2} f - \fint_{B_1} f\Big| < +\infty,$$

We shall prove the following law product.

Proposition 3. *Let $g \in BMO_F \cap L^p$, with $1 \leq p < \infty$, and $f \in LMO_F \cap L^\infty$. Then $fg \in BMO_F \cap L^p$ and*

$$\|fg\|_{BMO_F} \leq C\|f\|_{LMO_F\cap L^\infty} \|g\|_{BMO_F\cap L^p},$$

where C is independent of f and g.

Proof. In view Remark 3 we will consider throughout the proof B a ball of radius $r < \frac{1}{4}$. We start with writing the following plain identity

$$fg - \fint_B fg = f\Big(g - \fint_B g\Big) + \Big(\fint_B g\Big)\Big(f - \fint_B f\Big) + \fint_B \Big\{f\Big(\big(\fint_B g\big) - g\Big)\Big\}, \tag{19}$$

which gives in turn

$$\begin{aligned}\fint_B \Big|fg - \fint_B fg\Big| &\leq 2\fint_B |f|\Big|g - \fint_B g\Big| + \Big|\fint_B g\Big| \fint_B \Big|f - \fint_B f\Big| \\ &\lesssim \|f\|_{L^\infty}\|g\|_{BMO} + \Big|\fint_B g\Big| \fint_B \Big|f - \fint_B f\Big|. \end{aligned}\tag{20}$$

We denote by $\hat{B}$ the ball which is concentric to B and whose radius is equal to 1. According to the definition of the second part of the BMO_F norm we get

$$\begin{aligned}\Big|\fint_B g\Big| &\leq \Big|\fint_B g - \fint_{\hat{B}} g\Big| + \Big|\fint_{\hat{B}} g\Big| \\ &\lesssim F(1-\ln r)\|g\|_{BMO_F} + \|g\|_{L^p}. \end{aligned}\tag{21}$$

It follows that

$$\Big|\fint_B g\Big| \fint_B \Big|f - \fint_B f\Big| \lesssim \|g\|_{BMO_F\cap L^p}\|f\|_{LMO_F}.$$

Inserting this in (20) we find

$$\|fg\|_{BMO} \leq C\|g\|_{BMO_F\cap L^p}\|f\|_{LMO_F\cap L^\infty}.$$

For the second term of the BMO_F-norm we will make use of the following identity

$$\begin{aligned}\fint_{B_2} fg - \fint_{B_1} fg &= \fint_{B_2} f\Big(g - \fint_{B_2} g\Big) + \Big(\fint_{B_2} f\Big)\Big(\fint_{B_2} g - \fint_{B_1} g\Big) \\ &+ \Big(\fint_{B_1} g\Big)\Big(\fint_{B_2} f - \fint_{B_1} f\Big) + \fint_{B_1} f\Big(\big(\fint_{B_1} g\big) - g\Big). \end{aligned}\tag{22}$$

Since F is larger than 1 we may write

$$\frac{|\fint_{B_2} fg - \fint_{B_1} fg|}{F\Big(\dfrac{1-\ln r_2}{1-\ln r_1}\Big)} \leq \text{ I+II+III+IV},$$

where

$$\begin{aligned}
\mathrm{I} &\triangleq \Big|\fint_{B_2} f\big(g-\fint_{B_2} g\big)\Big|,\\
\mathrm{II} &\triangleq \frac{|\fint_{B_2} f||\fint_{B_2} g-\fint_{B_1} g|}{F\Big(\dfrac{1-\ln r_2}{1-\ln r_1}\Big)},\\
\mathrm{III} &\triangleq \Big|\fint_{B_1} g\Big|\Big|\fint_{B_2} f-\fint_{B_1} f\Big|,\\
\mathrm{IV} &\triangleq \Big|\fint_{B_1} f(g-\fint_{B_1} g)\Big|.
\end{aligned}$$

It is clear that I and IV are bounded by $\|f\|_{L^\infty}\|g\|_{BMO}$ and II is bounded by $\|f\|_{L^\infty}\|g\|_{BMO_F}$. It remains to estimate III. Reproducing the same argument used in (21) we get

$$\begin{aligned}
\mathrm{III} &\leq \|g\|_{BMO_F\cap L^p} F(1-\ln r_1)\Big|\fint_{B_2} f-\fint_{B_1} f\Big|\\
&\leq \|g\|_{BMO_F\cap L^p}\|f\|_{LMO_F}.
\end{aligned}$$

This completes the proof of the proposition. □

The next proposition deals with some useful properties of the LMO_F space.

Proposition 4. (i) *The $LMO_F\cap L^\infty$ space is an algebra. More precisely, there exists an absolute constant C such that for any $f,g\in LMO_F\cap L^\infty$ one has*

$$\|fg\|_{LMO_F\cap L^\infty}\leq C\big(\|f\|_{L^\infty}\|g\|_{LMO_F}+\|g\|_{L^\infty}\|f\|_{LMO_F}\big).$$

(ii) *Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be an entire real-function, which vanishes at 0. For any real-valued function u in $LMO_F\cap L^\infty$, the function $f\circ u$ belongs to the same space. Moreover, there exists a positive constant C and an entire real-function g such that we have*

$$\|f\circ u\|_{LMO_F\cap L^\infty}\leq C\|u\|_{LMO_F}\, g\big(\|u\|_{L^\infty}\big).$$

(iii) *For any $s>0$ we have the embedding $C^s\hookrightarrow LMO_F$, that is, there exits $C>0$ such that for any $f\in C^s$,*

$$\|f\|_{LMO_F}\leq C\|f\|_{C^s}.$$

Proof (**i**) Making appeal to inequality (20) we get

$$\begin{aligned}
F(1-\ln r)\fint_B\Big|fg-\fint_B fg\Big| &\leq 2F(1-\ln r)\fint_B |f|\Big|g-\fint_B g\Big|+F(1-\ln r)\Big|\fint_B g\Big|\fint_B\Big|f-\fint_B f\Big|\\
&\leq 2\|f\|_{L^\infty}\|g\|_{LMO_F}+\|g\|_{L^\infty}\|f\|_{LMO_F}.
\end{aligned}$$

Likewise, as $r_2\leq r_1$ and F is a nondecreasing function, we immediately deduce from identity (22) that

$$\begin{aligned}
F(1-\ln r_1)\Big|\fint_{B_2} fg-\fint_{B_1} fg\Big| &\leq F(1-\ln r_2)\fint_{B_2}|f|\Big|g-\fint_{B_2} g\Big|\\
&+ F(1-\ln r_1)\fint_{B_2}|f|\Big|\fint_{B_2} g-\fint_{B_1} g\Big|\Big)\\
&+ F(1-\ln r_1)\Big(\fint_{B_1}|g|\Big|\fint_{B_2} f-\fint_{B_1} f\Big|+\fint_{B_1}|f|\Big|g-\fint_{B_1} g\Big|\Big)\\
&\leq 2\|f\|_{L^\infty}\|g\|_{LMO_F}+2\|g\|_{L^\infty}\|f\|_{LMO_F}.
\end{aligned}$$

This completes the proof of the assertion (i).
(**ii**) By definition, there exists a sequence $(a_n)_n\subset\mathbb{R}$ such that for all $x\in\mathbb{R}$,

$$f(x)=\sum_{n\geq1}a_nx^n$$

and thus

$$f\circ u(x)=\sum_{n=1}^{\infty}a_n u^n(x).$$

Consequently

$$\|f\circ u\|_{LMO_F\cap L^\infty}\le\sum_{n=1}^{\infty}|a_n|\|u^n\|_{LMO_F\cap L^\infty}.$$

According to (**i**) and using the induction principle, we infer that for all $n\ge 2$ one has,

$$\|u^n\|_{LMO_F\cap L^\infty}\le C^{n-1}\|u\|_{LMO_F}\|u\|_{L^\infty}^{n-1}.$$

Therefore,

$$\begin{aligned}
\|f\circ u\|_{LMO_F\cap L^\infty} &\le \|u\|_{LMO_F}\sum_{n=1}^{\infty}|a_n|C^{n-1}\|u\|_{L^\infty}^{n-1}\\
&\le \|u\|_{LMO_F}\sum_{n=0}^{\infty}|a_{n+1}|C^{n}\|u\|_{L^\infty}^{n}\\
&\triangleq \|u\|_{LMO_F}\,g\big(\|u\|_{L^\infty}\big).
\end{aligned}$$

(**iii**) Using the definition of the Hölder space, we can write

$$\begin{aligned}
F(1-\ln r)\fint_B\big|f-\fint_B f\big| &\le F(1-\ln r)\fint_B\fint_B\big|f(x)-f(y)\big|dxdy\\
&\lesssim F(1-\ln r)r^s\|f\|_{C^s}\\
&\lesssim \|f\|_{C^s}.
\end{aligned}$$

The last inequality follows from (11) which implies that

$$\lim_{r\to 0}F(1-\ln r)r^s=0.$$

The second term of the norm can be handled exactly as the first one. For $x_0\in B_1\cap B_2$ we have

$$\begin{aligned}
F(1-\ln r_1)\Big|\fint_{B_2}f-\fint_{B_1}f\Big| &\le F(1-\ln r_2)\fint_{B_2}\big|f(x)-f(x_0)\big|dx\\
&+ F(1-\ln r_1)\fint_{B_1}\big|f(x)-f(x_0)\big|dx\\
&\le \big(F(1-\ln r_2)r_2^s+F(1-\ln r_1)r_1^s\big)\|f\|_{C^s},
\end{aligned}$$

which is bounded for the same reason as before.

3. Compressible transport model

We focus in this section on the study of the persistence regularity of the initial data measured in BMO_F space for the following compressible transport model:

$$\begin{cases}\partial_t f+v\cdot\nabla f+f\,\mathrm{div}\,v=0;\\ f_{|t=0}=f_0.\end{cases}\tag{23}$$

This model describes the vorticity dynamics for the system (E.C) and the advection is governed by a compressible velocity which not necessary in the Lipschitz class. According to the inequality (6) the velocity belongs to the log-Lipschitz class and as it was revealed in the paper [**?**] the solution in the incompressible case may exhibit a loss of regularity in the classical spaces like Sobolev and Hölder spaces. This possible loss cannot occur in the $LBMO$ space as it was recently proved in [6]. Here we shall extend this latter result to the model (23) and we will see later an application for the incompressible limit problem. Before stating our main result we will recall the log-Lipschitz norm:

$$\|v\|_{LL} = \sup_{0<|x-y|<1} \frac{|v(x)-v(y)|}{|x-y|\log\frac{e}{|x-y|}} < \infty$$

Our result reads as follows.

Theorem 2. *Let v be a smooth vector field and f be a smooth solution of the system (23). Then, there exits an absolute constant $C>0$ such that for every $1\le p\le\infty$,*

$$\|f(t)\|_{BMO_F\cap L^p} \le C\|f_0\|_{BMO_F\cap L^p} e^{C\|\operatorname{div} v\|_{L^1_t L^\infty}} F\big(e^{CV(t)}\big)\Big(1+F\big(e^{CV(t)}\big)\|\operatorname{div} v\|_{L^1_t(LMO_F\cap L^\infty)}\Big).$$

with

$$V(t) \triangleq \int_0^t \|v(\tau)\|_{LL} d\tau.$$

Remark 4. *When the velocity is divergence-free the estimate of the preceding theorem becomes*

$$\|f(t)\|_{BMO_F\cap L^p} \le C\|f_0\|_{BMO_F\cap L^p} F\big(e^{CV(t)}\big). \tag{24}$$

Therefore, Theorem 2 recovers the result stated in [6] for $F(x) = 1+\ln(x)$

$$\|f(t)\|_{LBMO\cap L^p} \quad\le\quad C\|f_0\|_{LBMO\cap L^p}\Big(1+\int_0^t \|v(\tau)\|_{LL} d\tau\Big).$$

As well, we mention that a similar estimate to (24) has been established in [5] in the incompressible framework for some $L^\alpha mo_F$ space.

To prove this result, we shall use the same approach of [5] and [6]. However the lack of the incompressibility brings more technical difficulties. Before going further into the details we should point out that the solution of the system (23) has, as we will see later, the following structure

$$f(t,x) = f_0(\psi^{-1}(t,x)) \exp\Big(-\int_0^t (\operatorname{div} v)(\tau,\psi(\tau,\psi^{-1}(t,x)))d\tau\Big).$$

Where ψ is the flow associated to the vector field v, that is, the solution of the differential equation,

$$\partial_t\psi(t,x) = v(t,\psi(t,x)), \quad \psi(0,x) = x.$$

It turns out that the study of the propagation in the BMO_F space returns to the study of the composition by the flow. Based on that, we shall firstly examine the regularity of the flow map, discuss its left composition with the elements of BMO_F and finally give the proof of Theorem 2.

3.1. **The regularity of the flow map.** Although the vector field v is not Lipschitz in our context, we still have existence and uniqueness of the flow but a loss of regularity may occur. In fact ψ is not necessary Lipschitz but belongs to the class C^{s_t} with $s_t<1$, as indicated in the lemma below. For more details see for instance [7] and [23].

Lemma 4. *Let v be a smooth vector field on $\mathbb{R}^2$ and ψ its flow . Then for all $t\in\mathbb{R}_+$, we have*

$$|x_1-x_2| < e^{-\beta(t)} \Longrightarrow |\psi^{\pm1}(t,x_1)-\psi^{\pm1}(t,x_2)| \le e|x_1-x_2|^{\frac{1}{\beta(t)}}.$$

where $\psi^1=\psi$ and ψ^{-1} is the inverse of ψ. The function $\beta(t)$ is given by

$$\beta(t) = \exp\Big(\int_0^t \|v\|_{LL} d\tau\Big).$$

As a consequence, we obtain the following lemma which is with an extreme importance in the proof of the composition result. Its proof can be found in [6].

Lemma 5. *Under the assumptions of Lemma 4 and for $r \le \exp(-\beta(t))$ we have*

$$4\psi(B(x_0,r)) \subset B(\psi(x_0), g_\psi(r)),$$

where

$$g_\psi(r) \triangleq 4er^{\frac{1}{\beta(t)}}. \tag{25}$$

In particular

$$\sup\Big\{\frac{1-\ln g_\psi(r)}{1-\ln r}, \frac{1-\ln r}{1-\ln g_\psi(r)}\Big\} \lesssim 1+\beta(t). \tag{26}$$

The following inequality will be frequently used in the rest of this paper, see for instance [7].

Lemma 6. *Let ψ be the flow associated to a smooth vector field v. Then for all $t\in\mathbb{R}_+$*

$$e^{-\int_0^t \|\mathrm{div}\, v(\tau)\|_{L^\infty}d\tau} \le |J_{\psi_t^{\pm 1}}(x)| \le e^{\int_0^t \|\mathrm{div}\, v(\tau)\|_{L^\infty}d\tau} \quad \forall x\in\mathbb{R}^2.$$

Where $J_{\psi_t}(t,x)$ is the Jacobian of $\psi(t,x)$.

3.2. **Composition in the BMO_F space.** The problem of the composition in the BMO space can be easily solved when ψ is a bi-Lipschitz map which is unfortunately not necessarily verified in our case. Such difficulty could in general induce a losing regularity but as we will see we can face up this loss by working in a suitable space and replace BMO space with the BMO_F spaces. We will be also led to deal with another technical difficulty linked to the fact that ψ is no longer measure-preserving map. Our result is the following,

Theorem 3. *There exists a positive constant C such that, for any function f taken in $BMO_F\cap L^p$, with $1\le p\le\infty$ and for ψ the flow associated to a smooth vector field v, we have*

$$\|f\circ\psi\|_{BMO_F\cap L^p} \le C\|f\|_{BMO_F\cap L^p} F\Big(e^{C\|v\|_{L^1_t LL}}\Big)e^{C\|\mathrm{div}\, v\|_{L^1_t L^\infty}}.$$

Proof. We know that, by a change of variable, the composition in the L^p space gives

$$\|f\circ\psi\|_{L^p} \le \|J_{\psi^{-1}}\|_{L^\infty}^{\frac{1}{p}}\|f\|_{L^p}.$$

The composition in the BMO_F space is more subtle and we shall use the idea of [6]. In fact, the proof is divided into two steps: in the first one we deal with the BMO term of the norm and in the second we consider the other term.

• **Step 1: Persistence of the BMO regularity.** We shall start with the persistence of the first part of the norm BMO_F. For this purpose we distinguish two cases depending whether the radius r is small or not.

<u>**Case 1**</u> : $r < (4e)^{-\beta(t)}\min\big\{1, \frac{1}{\|J_\psi\|_{L^\infty}}\big\}$. According to the definition (25) this condition implies

$$g_\psi(r) < 1.$$

We denote by $\tilde B$ the ball of center $\psi(x_0)$ and radius $g_\psi(r)$. It is easily seen that

$$⨍_B\Big|f\circ\psi - ⨍_B f\circ\psi\Big| \quad\le\quad 2⨍_B\Big|f\circ\psi - ⨍_{\tilde B} f\Big|.$$

Then by a change of variable one has

$$⨍_B\Big|f\circ\psi - ⨍_B f\circ\psi\Big| \quad\le\quad \frac{2\|J_{\psi^{-1}}\|_{L^\infty}}{|B|}\int_{\psi(B)}\Big|f - ⨍_{\tilde B} f\Big|dx.$$

At this stage the strategy consists in the partition of the open set $\psi(B)$ into countable balls with variable sizes and to try to measure their interactions with the biggest ball $\tilde B$. For this goal we shall use Whitney covering lemma [30] which asserts in our case the existence of a collection of countable open balls $(O_k)_k$ such that :

- The collection of double balls is a bounded covering :
$$\psi(B)\subset\bigcup_k 2O_k.$$
- The collection is disjoint and for all k,
$$O_k\subset\psi(B).$$
- The Whitney property is verified: the radius r_k of O_k satisfies
$$r_k\approx d(O_k,\psi(B)^c).$$

So by the first property we may write

$$\begin{aligned}\fint_B\Big|f\circ\psi-\fint_B f\circ\psi\Big|dx &\lesssim \frac{\|J_{\psi^{-1}}\|_{L^\infty}}{|B|}\sum_j|O_j|\fint_{2O_j}\Big|f-\fint_{\tilde B}f\Big|\\ &\lesssim \|J_{\psi^{-1}}\|_{L^\infty}\big(I_1+I_2\big),\end{aligned}$$

where

$$\mathrm{I}_1\triangleq\frac{1}{|B|}\sum_j|O_j|\fint_{2O_j}\Big|f-\fint_{2O_j}f\Big|$$

and

$$\mathrm{I}_2\triangleq\frac{1}{|B|}\sum_j|O_j|\Big|\fint_{2O_j}f-\fint_{\tilde B}f\Big|.$$

Using the fact

$$\sum_j|O_j|\le|\psi(B)|\le\|J_\psi\|_{L^\infty}|B|,\tag{27}$$

we immediately deduce that

$$\mathrm{I}_1\lesssim\|J_\psi\|_{L^\infty}\|f\|_{BMO}.$$

According to Lemma 5 we have $4O_j\subset\tilde B$. In addition, as $g_\psi(r)<1$ and in view of the definition of the BMO_F-norm, we infer that

$$\begin{aligned}\mathrm{I}_2 &\lesssim \frac{1}{|B|}\sum_j|O_j|F\Big(\frac{1-\ln 2r_j}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}\\ &\lesssim \frac{1}{|B|}\sum_j|O_j|F\Big(\frac{1-\ln r_j}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}.\end{aligned}$$

From (27) we get $r_j\le\|J_\psi\|_{L^\infty}^{1/2}r$ for all j. Set

$$h(r)\triangleq r\max\big\{1,\|J_\psi\|_{L^\infty}\big\},\quad U_k\triangleq\sum_{e^{-k-1}h(r)<r_j\le e^{-k}h(r)}|O_j|,\quad k\in\mathbb{N}.$$

Hence as F is non-decreasing we may write

$$\begin{aligned}\mathrm{I}_2 &\lesssim \frac{1}{|B|}\sum_{k\ge0}U_kF\Big(\frac{k+2-\ln h(r)}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}\\ &\lesssim \frac{1}{|B|}\sum_{k\ge0}U_kF\Big(\frac{k+2-\ln r}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}.\end{aligned}$$

Let N be a fixed positive integer that will be carefully chosen later. We split the preceding sum into two parts

$$\begin{aligned}\mathrm{I}_2 &\lesssim \bigg(\frac{1}{|B|}\sum_{k\le N}U_kF\Big(\frac{k+2-\ln r}{1-\ln g_\psi(r)}\Big)+\frac{1}{|B|}\sum_{k>N}U_kF\Big(\frac{k+2-\ln r}{1-\ln g_\psi(r)}\Big)\bigg)\|f\|_{BMO_F}\\ &\triangleq \big(\mathrm{I}_{2,1}+\mathrm{I}_{2,2}\big)\|f\|_{BMO_F}.\end{aligned}$$

Since $\sum U_k \leq \|J_\psi\|_{L^\infty}|B|$ and F is non-decreasing then

$$\mathrm{I}_{2,1} \leq \|J_\psi\|_{L^\infty} F\Big(\frac{N+2-\ln r}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}. \tag{28}$$

To estimate $\mathrm{I}_{2,2}$ we need the following bound of U_k whose proof will be given in Lemma 8 of the Appendix.

$$U_k \lesssim \big(1+\|J_\psi\|_{L^\infty}\big)^2 e^{\frac{-k}{\beta(t)}} r^{1+\frac{1}{\beta(t)}} \quad \forall k \geq \beta(t).$$

Therefore for N taken larger then $\beta(t)$ we have

$$\mathrm{I}_{2,2} \quad \lesssim \quad \big(1+\|J_\psi\|_{L^\infty}\big)^2 \sum_{k>N} e^{-\frac{k}{\beta(t)}} r^{\frac{1}{\beta(t)}-1} F\Big(\frac{k+2-\ln r}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}.$$

Inequality (12) from Remark 2 gives

$$\mathrm{I}_{2,2} \quad \lesssim \quad \big(1+\|J_\psi\|_{L^\infty}\big)^2 e^{-\frac{N}{\beta(t)}} \beta(t) r^{\frac{1}{\beta(t)}-1} F\Big(\frac{N+2-\ln r}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}. \tag{29}$$

Combining this estimate with (28) we obtain

$$\mathrm{I}_2 \lesssim \big(1+\|J_\psi\|_{L^\infty}\big)^2 \Big(1+e^{\frac{-N}{\beta(t)}}\beta(t) r^{\frac{1}{\beta(t)}-1}\Big) F\Big(\frac{N+2-\ln r}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}.$$

Taking $N = \big[\beta(t)(\beta(t)-\ln r)\big]+1$ we get

$$\begin{aligned}
\mathrm{I}_2 \quad &\lesssim \quad \big(1+\|J_\psi\|_{L^\infty}\big)^2 \big(1+e^{-\beta(t)} e^{-\frac{1}{\beta(t)}(1-\ln r)}\beta(t)\big) F\Big(\frac{\beta(t)(\beta(t)-\ln r)+3-\ln r}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F} \\
&\lesssim \quad \big(1+\|J_\psi\|_{L^\infty}\big)^2 F\Big(\frac{\beta(t)^2+3-(1+\beta(t))\ln r}{1-\ln g_\psi(r)}\Big)\|f\|_{BMO_F}.
\end{aligned}$$

Where we have used in the last inequality the fact that $\sup_{\beta>1}\beta e^{-\beta} < 1$. Furthermore, from estimate (26) we have

$$\begin{aligned}
F\Big(\frac{\beta(t)^2+2-(1+\beta(t))\ln r}{1-\ln g_\psi(r)}\Big) \quad &\lesssim \quad F\Big((1+\beta(t))\frac{\beta(t)^2+3-(1+\beta(t))\ln r}{1-\ln r}\Big) \\
&\lesssim \quad F\big((1+\beta(t))^3\big) \\
&\lesssim \quad F(\beta^3(t)).
\end{aligned}$$

Hence,

$$\mathrm{I}_2 \lesssim \big(1+\|J_\psi\|^2_{L^\infty}\big) F\big(\beta^3(t)\big)\|f\|_{BMO_F}.$$

Putting together the previous estimates gives

$$⨍_B \Big|f\circ\psi - ⨍_B f\circ\psi\Big| \quad \leq \quad \|J_{\psi^{-1}}\|_{L^\infty}\big(1+\|J_\psi\|^2_{L^\infty}\big) F\big(\beta^3(t)\big)\|f\|_{BMO_F}.$$

According to Lemma 6 we find

$$⨍_B \Big|f\circ\psi - ⨍_B f\circ\psi\Big| \leq Ce^{C\|\mathrm{div}\ v\|_{L^1_t L^\infty}} F\big(\beta^3(t)\big)\|f\|_{BMO_F}. \tag{30}$$

Let us now move to the second case.
<u>**Case 2**</u> : $1 \geqslant r \geq (4e)^{-\beta(t)} \min\big\{1, \frac{1}{\|J_\psi\|_{L^\infty}}\big\}$. Under this assumption we can easily check that

$$|\ln r| \lesssim \beta(t) + \big|\ln\|J_\psi\|_{L^\infty}\big|. \tag{31}$$

By a change of variable we can write

$$\begin{aligned}
\fint_B \Big| f\circ\psi - \fint_B f\circ\psi \Big| &\leq 2\fint_B |f\circ\psi| \\
&\lesssim \frac{1}{|B|}\int_{\psi(B)} |f(x)||J_{\psi^{-1}}(x)|dx \\
&\lesssim \frac{1}{|B|}\sum_j |O_j|\|J_{\psi^{-1}}\|_{L^\infty}\fint_{2O_j}|f| \\
&\lesssim \|J_{\psi^{-1}}\|_{L^\infty}\Big(\frac{1}{|B|}\sum_j |O_j|\fint_{2O_j}\Big|f-\fint_{2O_j} f\Big| + \frac{1}{|B|}\sum_j \Big|\int_{2O_j} f\Big|\Big) \\
&\lesssim \|J_{\psi^{-1}}\|_{L^\infty}\Big(\|J_\psi\|_{L^\infty}\|f\|_{BMO} + \mathrm{I}_1 + \mathrm{I}_2\Big),
\end{aligned}$$

where

$$\mathrm{I}_1 \triangleq \frac{1}{|B|}\sum_{j\backslash r_j>\frac14}\Big|\int_{2O_j} f\Big|,$$

and

$$\mathrm{I}_2 \triangleq \frac{1}{|B|}\sum_{j\backslash r_j\leq\frac14}|O_j|\Big|\fint_{2O_j} f\Big|.$$

Hölder inequality implies that

$$\begin{aligned}
\mathrm{I}_1 &\leq \frac{1}{|B|}\sum_{j\backslash r_j>\frac14}|O_j|^{1-\frac1p}\|f\|_{L^p} \\
&\lesssim \frac{1}{|B|}\sum_j |O_j|\|f\|_{L^p} \\
&\lesssim \|J_\psi\|_{L^\infty}\|f\|_{L^p}.
\end{aligned}$$

In order to estimate the term I_2, we consider a collection of open balls $(\tilde{O}_j)_j$ such that, for all j in $\mathbb{N}$, $\tilde{O}_j$ is concentric to O_j and of radius equal to 1. Then, as $r_j \leq \frac14$ we have $4O_j \subset \tilde{O}_j$ and thus using the definition of the BMO_F-norm gives

$$\begin{aligned}
\mathrm{I}_2 &\lesssim \frac{1}{|B|}\sum_{j\backslash r_j\leq\frac14}|O_j|\Big|\fint_{2O_j} f - \fint_{\tilde{O}_j} f\Big| + \frac{1}{|B|}\sum_{j\backslash r_j\leq\frac14}|O_j|\Big|\fint_{\tilde{O}_j} f\Big| \\
&\lesssim \frac{1}{|B|}\sum_{j\backslash r_j\leq\frac14}|O_j|F(1-\ln 2r_j)\|f\|_{BMO_F} + \|J_\psi\|_{L^\infty}\|f\|_{L^p}.
\end{aligned}$$

We set

$$V_k \triangleq \sum_{\substack{j\\ e^{-k-1}\leq 4r_j\leq e^{-k}}} |O_j|.$$

Then,

$$\frac{1}{|B|}\sum_{r_j\leq\frac14}|O_j|F(1-\ln 2r_j) \leq \frac{1}{|B|}\sum_{k\geq 0} V_k F(k+4).$$

Fix $N\in\mathbb{N}^\star$ and split the last sum into two parts according to $k\geq N$ and $k>N$ gives

$$\frac{1}{|B|}\sum_{r_j\leq\frac14}|O_j|F(1-\ln 2r_j) \quad\leq\quad \frac{1}{|B|}\sum_{k\leq N} V_k F(N+4) + \frac{1}{|B|}\sum_{k>N} V_k F(k+4).$$

For $N \geq \beta(t)$, we may use Lemma 8 and inequality (12) leading to

$$\begin{aligned}\frac{1}{|B|}\sum_j |O_j| F(1-\ln 2r_j) &\lesssim \|J_\psi\|_{L^\infty}\Big(F(N+4)+\sum_{k>N} e^{-\frac{k}{\beta(t)}} r^{-1} F(k+4)\Big)\\ &\lesssim \|J_\psi\|_{L^\infty}\big(1+e^{-\frac{N}{\beta(t)}}\beta(t) r^{-1}\big)F(N+4).\end{aligned}$$

We choose $N = \big[\beta(t)(\beta(t)-\ln r)\big]$ and by using (31) we obtain

$$\frac{1}{|B|}\sum_j |O_j| F(1-\ln 2r_j) \lesssim \|J_\psi\|_{L^\infty} F\big(1+\beta(t)^2+\beta(t)\big|\ln \|J_\psi\|_{L^\infty}\big|\big).$$

Putting together the previous estimates gives

$$\begin{aligned}\fint_B \Big| f\circ\psi - \fint_B f\circ\psi\Big| &\leq \|J_{\psi^{-1}}\|_{L^\infty}\|J_\psi\|_{L^\infty}\|f\|_{BMO_F\cap L^p} F\big(\beta(t)^2+\beta(t)\big|\ln\|J_\psi\|_{L^\infty}\big|\big)\\ &\leq Ce^{C\|\mathrm{div}\, v\|_{L^1_t L^\infty}}\|f\|_{BMO_F\cap L^p} F\Big(e^{C\|v\|_{L^1_t LL}}\|\mathrm{div}\, v\|_{L^1_t L^\infty}\Big).\end{aligned}$$

Combining this estimate with (30) we obtain

$$\|f\circ\psi\|_{BMO} \lesssim C\|f\|_{BMO_F\cap L^p} e^{C\|\mathrm{div}\, v\|_{L^1_t L^\infty}} F\Big(e^{C\|v\|_{L^1_t LL}}\|\mathrm{div}\, v\|_{L^1_t L^\infty}\Big).$$

In view of the polynomial growth condition of F seen in (11) we get

$$\|f\circ\psi\|_{BMO} \leq C\|f\|_{BMO_F\cap L^p} e^{C\int_0^t \|\mathrm{div} v(\tau)\|_{L^\infty} d\tau} F\big(e^{C\|v\|_{L^1_t LL}}\big).$$

Now we shall move to the treatment of the second part of the BMO_F norm.

• **Step 2: Estimate of the second part of the BMO_F norm**. This will be done in a similar way to the first part. Denote $B_i = B(x_i, r_i)$ and $\tilde{B}_i = B(x_i, g_\psi(r_i))$ for $i \in \{1,2\}$, with $2B_2 \subset B_1$ and $r_1 < 1$. Set

$$J \triangleq \frac{\big|\fint_{B_2} f\circ\psi - \fint_{B_1} f\circ\psi\big|}{F\Big(\dfrac{1-\ln r_2}{1-\ln r_1}\Big)},$$

We have three cases to discuss:
<u>**Case 1**</u> : $r_1 \leq (4e)^{-\beta(t)} \min\big\{1, \frac{1}{\|J_\psi\|_{L^\infty}}\big\}$. Since the denominator of the quantity J is larger than one, we may write

$$J \leq J_1 + J_2 + J_3.$$

Where

$$\begin{aligned} J_1 &\triangleq \Big|\fint_{B_2} f\circ\psi - \fint_{\tilde{B}_2} f\Big| + \Big|\fint_{B_1} f\circ\psi - \fint_{\tilde{B}_1} f\Big|,\\ J_2 &\triangleq \frac{\big|\fint_{\tilde{B}_2} f - \fint_{2\tilde{B}_1} f\big|}{F\Big(\dfrac{1-\ln r_2}{1-\ln r_1}\Big)},\\ J_3 &\triangleq \Big|\fint_{\tilde{B}_1} f - \fint_{2\tilde{B}_1} f\Big|.\end{aligned}$$

The treatment of J_1 will be exactly the same as for the case 1 from step 1. For J_2, by definition of the second part of the BMO_F norm, we have

$$J_2 \leq \frac{F\Big(\frac{1-\ln g_\psi(r_2)}{1-\ln g_\psi(r_1)}\Big)}{F\Big(\dfrac{1-\ln r_2}{1-\ln r_1}\Big)}\|f\|_{BMO_F}.$$

According to the inequality (26), we get

$$\begin{aligned} F\Big(\frac{1-\ln g_\psi(r_2)}{1-\ln g_\psi(r_1)}\Big) &= F\Big(\frac{1-\ln r_1}{1-\ln g_\psi(r_1)}\quad\frac{1-\ln r_2}{1-\ln r_1}\quad\frac{1-\ln g_\psi(r_2)}{1-\ln r_2}\Big)\\ &\lesssim F\Big((1+\beta(t))^2\frac{1-\ln r_2}{1-\ln r_1}\Big). \end{aligned}$$

This gives in view of Definition 1,

$$J_2 \lesssim F\big(\beta^2(t)\big)\|f\|_{BMO_F}.$$

Concerning J_3 we use the inequality (10) to get

$$J_3 \lesssim \|f\|_{BMO}.$$

Case 2 : $r_2 \geqslant (4e)^{-\beta(t)}\min\big\{1, \frac{1}{\|J_\psi\|_{L^\infty}}\big\}$. As F is larger than 1, we write

$$J \le \fint_{B_2}|f\circ\psi| + \fint_{B_1}|f\circ\psi|.$$

Both terms can be handled as in the analysis of the case 2 from step 1.

Case 3 : $r_1 \geqslant (4e)^{-\beta(t)}\min\big\{1, \frac{1}{\|J_\psi\|_{L^\infty}}\big\}$ and $r_2 \le (4e)^{-\beta(t)}\min\big\{1, \frac{1}{\|J_\psi\|_{L^\infty}}\big\}$. We decompose J as follows:

$$\begin{aligned} J &\le \fint_{B_2}\Big|f\circ\psi - \fint_{\tilde B_2} f\Big| + \frac{|\fint_{\tilde B_2} f|}{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big)} + \fint_{B_1}|f\circ\psi|\\ &\triangleq J_1+J_2+J_3. \end{aligned}$$

Let $\tilde B_2'$ the ball of center $\psi(x_2)$ and radius 1. Then

$$\begin{aligned} J_2 &\le \frac{|\fint_{\tilde B_2} f - \fint_{\tilde B_2'} f| + |\fint_{\tilde B_2'} f|}{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big)}\\ &\le \frac{F(1-\ln g_\psi(r_2))}{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big)}\|f\|_{BMO_F} + \|f\|_{L^p}. \end{aligned}$$

From the Definition 1 combined with the inequality (26), we obtain

$$\begin{aligned} \frac{F(1-\ln g_\psi(r_2))}{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big)} &= \frac{F\Big(\frac{1-\ln g_\psi(r_2)}{1-\ln r_2}\quad\frac{1-\ln r_2}{1-\ln r_1}\quad 1-\ln r_1\Big)}{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big)}\\ &\lesssim \frac{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\quad (1+\beta(t))(1-\ln r_1)\Big)}{F\Big(\frac{1-\ln r_2}{1-\ln r_1}\Big)}\\ &\lesssim F\big((1+\beta(t)(\beta+|\ln\|J_\psi\|_{L^\infty}|)\\ &\lesssim F(e^{C\|v\|_{L^1_tLL}})F\big(\|\mathrm{div}v\|_{L^1_tL^\infty}\big). \end{aligned}$$

The terms J_1 and J_3 can be handled in the same way as the cases 1 and 2 from step 1. The proof is now achieved.

□

Our next task is to study the composition in the space $LMO_F\cap L^\infty$. This will more easier than the the BMO_F space since we shall use in a crucial way the L^∞ norm.

Proposition 5. *There exists a positive constant C such that, for any function $f\in LMO_F\cap L^\infty$ and for ψ a flow associated to a smooth vector field v, we have*

$$\|(f\circ\psi)(t)\|_{LMO_F}\leq CF\big(e^{C\|v\|_{L^1_tLL}}\big)e^{C\|\mathrm{div}\, v\|_{L^1_tL^\infty}}\|f\|_{LMO_F\cap L^\infty}.$$

Proof. Identically to the proof of the Theorem 5, we will proceed in two steps; the first one concerns the first term of the norm and the second one is devoted to the other term.

• **Step1: Estimate of the first part of the norm.** One distinguishes two cases:

<u>**Case 1**</u> : $r\leq(4e)^{-\beta(t)}$. We may write

$$F(1-\ln r)\fint_B\Big|f\circ\psi-\fint_B f\circ\psi\Big|dx\quad\leq\quad 2F(1-\ln r)\fint_B\Big|f\circ\psi-\fint_{\tilde B}f\Big|dx.$$

Recall that $\tilde B$ is the ball of center $\psi(x_0)$ and radius $g_\psi(r)$. A change of variable gives

$$F(1-\ln r)\fint_B\Big|f\circ\psi-\fint_B f\circ\psi\Big|dx\quad\lesssim\quad\|J_{\psi^{-1}}\|_{L^\infty}\frac{F(1-\ln r)}{|B|}\int_{\psi(B)}\Big|f-\fint_{\tilde B}f\Big|dx.$$

Using the Whitney covering lemma used in the proof of the Theorem 5 we get

$$\begin{aligned}F(1-\ln r)\fint_B\Big|f\circ\psi-\fint_B f\circ\psi\Big|dx\quad&\lesssim\quad\|J_{\psi^{-1}}\|_{L^\infty}\frac{F(1-\ln r)}{|B|}\sum_j|O_j|\fint_{2O_j}\Big|f-\fint_{\tilde B}f\Big|\\&\lesssim\quad\|J_{\psi^{-1}}\|_{L^\infty}\big(\mathrm{I}_1+\mathrm{I}_2\big),\end{aligned}$$

where

$$I_1\triangleq\frac{F(1-\ln r)}{|B|}\sum_j|O_j|\fint_{2O_j}\Big|f-\fint_{2O_j}f\Big|,$$

and

$$\mathrm{I}_2\triangleq\frac{F(1-\ln r)}{|B|}\sum_j|O_j|\Big|\fint_{2O_j}f-\fint_{\tilde B}f\Big|.$$

In view of the the polynomial growth property of F, we have

$$\begin{aligned}\mathrm{I}_1\quad&=\quad\frac{1}{|B|}\sum_j|O_j|F\Big(\frac{1-\ln r}{1-\ln 2r_j}(1-\ln(2r_j))\Big)\fint_{2O_j}\Big|f-\fint_{2O_j}f\Big|\\&\lesssim\quad\frac{1}{|B|}\sum_j|O_j|F\Big(\frac{1-\ln r}{1-\ln(2r_j)}\Big)F(1-\ln(2r_j))\fint_{2O_j}\Big|f-\fint_{2O_j}f\Big|\\&\lesssim\quad\frac{1}{|B|}\sum_j|O_j|F\Big(\frac{1-\ln r}{1-\ln(2r_j)}\Big)\|f\|_{LMO_F}\end{aligned}$$

As $r_j\leq g_\psi(r)$ and by (26) one has

$$\begin{aligned}\frac{1-\ln r}{1-\ln 2r_j}\quad&\lesssim\quad\frac{1-\ln r}{1-\ln 2g_\psi(r)}\\&\lesssim\quad 1+\beta(t).\end{aligned}$$

Consequently we get in view of (27),

$$\mathrm{I}_1\lesssim\|J_\psi\|_{L^\infty}F\big(\beta(t)\big)\|f\|_{LMO_F}.$$

Since $4O_j \subset \tilde{B}$, $g_\psi(r) \leq 1$ and by the definition of the second part of the BMO_F-norm, I_2 can be estimated as follows

$$
\begin{aligned}
\mathrm{I}_2 &\lesssim \frac{1}{|B|}\sum_j |O_j| F\Big(\frac{1-\ln r}{1-\ln g_\psi(r)}\big(1-\ln g_\psi(r)\big)\Big)\Big|\fint_{2O_j} f - \fint_{\tilde{B}} f\Big| \\
&\lesssim \frac{1}{|B|}\sum_j |O_j| F\big(1+\beta(t)\big) F(1-\ln g_\psi(r))\Big|\fint_{2O_j} f - \fint_{\tilde{B}} f\Big| \\
&\lesssim \|J_\psi\|_{L^\infty} F\big(\beta(t)\big)\|f\|_{LMO_F}.
\end{aligned}
$$

<u>Case 2</u> : $1 \geqslant r \geq (4e)^{-\beta(t)}$. Under this assumption we have $|\ln r| \lesssim \beta(t)$ and then we can immediately deduce that

$$
\begin{aligned}
F(1-\ln r)\fint_B \Big|f\circ\psi - \fint_B f\circ\psi\Big| &\lesssim F\big(1+\beta(t)\big)\fint_B \big|f\circ\psi\big| \\
&\lesssim F\big(\beta(t)\big)\|f\|_{L^\infty}.
\end{aligned}
$$

Step 2: Estimate of the second part of the norm. Denote $B_i = B(x_i, r_i)$ and $\tilde{B}_i = (x_i, g_\psi(r_i))$ for $i \in \{1,2\}$ with $2B_2 \subset B_1$ and $r_1 < 1$. We shall estimate J defined by,

$$
J \triangleq F(1-\ln r_1)\Big|\fint_{B_2} f\circ\psi - \fint_{B_1} f\circ\psi\Big|.
$$

There are two cases to discuss depending on the size of r_1.
<u>Case 1</u> : $r_1 \leq (4e)^{-\beta(t)}$. We write

$$
J \leq J_1 + J_2 + J_3,
$$

where

$$
\begin{aligned}
J_1 &\triangleq F(1-\ln r_1)\Big(\Big|\fint_{B_2} f\circ\psi - \fint_{\tilde{B}_2} f\Big| + \Big|\fint_{B_1} f\circ\psi - \fint_{\tilde{B}_1} f\Big|\Big) \\
J_2 &\triangleq F(1-\ln r_1)\Big|\fint_{\tilde{B}_2} f - \fint_{2\tilde{B}_1} f\Big| \\
J_3 &\triangleq F(1-\ln r_1)\Big|\fint_{\tilde{B}_1} f - \fint_{2\tilde{B}_1} f\Big|.
\end{aligned}
$$

Reproducing the same arguments as for the case 1 from step 1 we can estimate J_1. For J_2, we use the polynomial growth property of F with the inequality (26) to get

$$
\begin{aligned}
J_2 &\leq F\Big(\frac{1-\ln r_1}{1-\ln 2g_\psi(r_1)}\big(1-\ln 2g_\psi(r_1)\big)\Big)\Big|\fint_{\tilde{B}_2} f - \fint_{2\tilde{B}_1} f\Big| \\
&\leq F\Big(\frac{1-\ln r_1}{1-\ln 2g_\psi(r_1)}\Big) F\big(1-\ln 2g_\psi(r_1)\big)\Big|\fint_{\tilde{B}_2} f - \fint_{2\tilde{B}_1} f\Big| \\
&\leq F\big(1+\beta(t)\big)\|f\|_{LMO_F}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
J_3 &\leq F\Big(\frac{1-\ln r_1}{1-\ln 2g_\psi(r_1)}\Big) F\big(1-\ln 2g_\psi(r_1)\big)\Big|\fint_{\tilde{B}_1} f - \fint_{2\tilde{B}_1} f\Big| \\
&\leq F\big(1+\beta(t)\big)\|f\|_{LMO_F}.
\end{aligned}
$$

<u>Case 2</u> : $r_1 \geqslant (4e)^{-\beta(t)}$. Since F is non-decreasing then,

$$
\begin{aligned}
J &\leq F(1-\ln r_1)\Big(\fint_{B_2} \big|f\circ\psi\big| + \fint_{B_1} \big|f\circ\psi\big|\Big) \\
&\lesssim F(\beta(t))\|f\|_{L^\infty}.
\end{aligned}
$$

This completes the proof Proposition 5. $\square$

Now we have all the necessary ingredients for the proof of Theorem 2

3.3. **Proof of Theorem 2.** We set $g(t,x) = f(t,\psi(t,x))$. Then, in view of (23), we see that g satisfies the following equation

$$\partial_t g(t,x) + (\mathrm{div}\, v)(t,\psi(t,x))g(t,x) = 0, \quad g(0,x) = f_0(x).$$

It follows that

$$\begin{aligned} g(t,x) &= f_0(x)e^{-\int_0^t (\mathrm{div}\, v)(\tau,\psi(\tau))d\tau} \\ &= f_0(x) + f_0(x)\Big(e^{-\int_0^t (\mathrm{div}\, v)(\tau,\psi(\tau))d\tau} - 1\Big). \end{aligned}$$

According to the law product stated in Proposition 3, we have

$$\|g(t)\|_{BMO_F\cap L^p} \le C\|f_0\|_{BMO_F\cap L^p}\Big(1 + \|e^{-\int_0^t (\mathrm{div}\, v)(\tau,\psi(\tau))d\tau} - 1\|_{LMO_F\cap L^\infty}\Big).$$

Therefore by applying the assertion (ii) of Proposition 4 to the function $x \longmapsto e^x - 1$, we get

$$\|g(t)\|_{BMO_F\cap L^p} \le C\|f_0\|_{BMO_F\cap L^p}\Big(1 + e^{C\|\mathrm{div}\, v\|_{L^1_t L^\infty}}\int_0^t \|(\mathrm{div}\, v)(\tau,\psi(\tau))\|_{LMO_F\cap L^\infty} d\tau\Big).$$

Furthermore, according to Proposition 5 we infer that

$$\|g(t)\|_{BMO_F\cap L^p} \le C\|f_0\|_{BMO_F\cap L^p}\Big(1 + e^{C\|\mathrm{div}\, v\|_{L^1_t L^\infty}} F\Big(e^{C\|v\|_{L^1_t LL}}\Big)\|\mathrm{div}\, v\|_{L^1_t(LMO_F\cap L^\infty)}\Big).$$

Finally, Theorem 3 gives

$$\|f(t)\|_{BMO_F\cap L^p} \le C\|f_0\|_{BMO_F\cap L^p} e^{C\|\mathrm{div}\, v\|_{L^1_t L^\infty}} F\Big(e^{C\|v\|_{L^1_t LL}}\Big)\Big(1 + F\big(e^{C\|v\|_{L^1_T LL}}\big)\|\mathrm{div}\, v\|_{L^1_t(LMO_F\cap L^\infty)}\Big).$$

This completes the proof of the theorem.

4. Some classical estimates

The aim of this section is to highlight two useful estimates for the system (E.C), those will be of great importance for obtaining a lower bound of the lifespan of the solution v_ε. In the first instance we shall recall classical energy estimates for the full system, afterwards, we lead a short discussion on Strichartz estimates for the wave operator.

4.1. **Energy estimates.** The following energy estimates are classical and for the proof we refer the reader for instance to [9, 15, 17].

Proposition 6. *Let $(v_\varepsilon, c_\varepsilon)$ be a smooth solution of* (E.C). *For $s > 0$ there exists a constant $C > 0$ such that*

$$\|(v_\varepsilon, c_\varepsilon)(t)\|_{H^s} \le C\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^s} e^{CV_\varepsilon(t)}, \quad \forall t \ge 0$$

with

$$V_\varepsilon(t) \triangleq \|\nabla v_\varepsilon\|_{L^1_t L^\infty} + \|\nabla c_\varepsilon\|_{L^1_t L^\infty}.$$

4.2. **Strichartz estimates.** The main interest of using Strichartz estimates is to deal with the ill-prepared data in the presence of the singular terms in $\frac{1}{\varepsilon}$. Actually, it has been shown that the average in time of the compressible and the acoustic parts, which are governed by a coupling non linear wave equations, disappear when the Mach number approaches zero. The details of this assumption has been discussed for instance in [15] and [16] for initial data in Besov spaces, but for the convenience of the reader we briefly outline the main arguments of the proof. The system (E.C) can be rewritten under the form

$$\begin{cases} \partial_t v_\varepsilon + \frac{1}{\varepsilon}\nabla c_\varepsilon = -v_\varepsilon.\nabla v_\varepsilon - \bar\gamma c_\varepsilon \nabla c_\varepsilon \triangleq f_\varepsilon \\ \partial_t c_\varepsilon + \frac{1}{\varepsilon}\mathrm{div}\, v_\varepsilon = -v_\varepsilon.\nabla c_\varepsilon - \bar\gamma c_\varepsilon \mathrm{div}\, v_\varepsilon \triangleq g_\varepsilon \\ (v_\varepsilon, c_\varepsilon)_{|t=0} = (v_{\varepsilon,0}, c_{\varepsilon,0}). \end{cases}$$

We denote by $\mathbb{Q}v_\varepsilon \triangleq \nabla\Delta^{-1}\mathrm{div}\, v_\varepsilon$ the compressible part of the velocity v_ε. Then the complex-valued functions

$$\Gamma_\varepsilon \triangleq \mathbb{Q}v_\varepsilon - i\nabla|\mathrm{D}|^{-1}c_\varepsilon \quad \text{and} \quad \Upsilon_\varepsilon \triangleq |\mathrm{D}|^{-1}\mathrm{div}\, v_\varepsilon + ic_\varepsilon$$

satisfy the following wave equations

$$(\partial_t + \frac{i}{\varepsilon}|\mathrm{D}|)\Gamma_\varepsilon = \mathbb{Q}f_\varepsilon - i\nabla|\mathrm{D}|^{-1}g_\varepsilon \tag{32}$$

and

$$(\partial_t + \frac{i}{\varepsilon}|\mathrm{D}|)\Upsilon_\varepsilon = |\mathrm{D}|^{-1}\mathrm{div}\, f_\varepsilon - ig_\varepsilon \tag{33}$$

with $|\mathrm{D}| = (-\Delta)^{\frac{1}{2}}$.

Now we can following Strichartz estimates whose proof can be found for instance in [4, 9, 13].

Lemma 7. *Let φ be a solution of the wave equation*

$$(\partial_t + \frac{i}{\varepsilon}|\mathrm{D}|^{-1})\varphi = G, \quad \varphi_{|t=0} = \varphi_0.$$

Then, there exists a constant $C > 0$ such that for all $T > 0$ and $2 < p \leq +\infty$,

$$\|\varphi\|_{L^r_T L^p} \leq C\varepsilon^{\frac{1}{4}-\frac{1}{2p}}\Big(\|\varphi_0\|_{\dot{B}^{\frac{3}{4}-\frac{3}{2p}}_{2,1}} + \int_0^t \|G(\tau)\|_{\dot{B}^{\frac{3}{4}-\frac{3}{2p}}_{2,1}} d\tau\Big),$$

with $r = 4 + \frac{8}{p-2}$.

As an application we get the following result.

Corollary 1. *Let $s > 0$ and $(v_{0,\varepsilon}, c_{0,\varepsilon})$ be a family in H^{2+s}. Then any solution of (E.C) defined in the time interval $[0, T]$ satisfies*

$$\|(\mathbb{Q}v_\varepsilon, c_\varepsilon)\|_{L^4_T L^\infty} \leq C_0^\varepsilon \varepsilon^{\frac{1}{4}}(1+T)e^{CV_\varepsilon(T)}. \tag{34}$$

Moreover, there exists a positive real number η which depends only on s such that

$$\|(\mathrm{div}\, v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_T B^{s/3}_{\infty,\infty}} \leq C_0^\varepsilon(1+T^{\frac{7}{4}})\varepsilon^\eta e^{CV_\varepsilon(T)}. \tag{35}$$

Where

$$V_\varepsilon(T) \triangleq \|\nabla v_\varepsilon\|_{L^1_T L^\infty} + \|\nabla c_\varepsilon\|_{L^1_T L^\infty},$$

and C_0^ε depends only on the quantity $\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^{2+s}}$ and with polynomial growth.

Proof. Applying Lemma 7 to the equation (32), we get

$$\|\Gamma_\varepsilon\|_{L^4_T L^\infty} \lesssim \varepsilon^{\frac{1}{4}}\Big(\|\Gamma^0_\varepsilon\|_{\dot{B}^{\frac{3}{4}}_{2,1}} + \int_0^T \|(\mathbb{Q}f_\varepsilon - i\nabla|\mathrm{D}|^{-1}g_\varepsilon)(\tau)\|_{\dot{B}^{\frac{3}{4}}_{2,1}} d\tau\Big).$$

Since $\mathbb{Q}$ and $\nabla|\mathrm{D}|^{-1}$ act continuously on the homogeneous Besov spaces, we get

$$\begin{aligned}\|\Gamma_\varepsilon\|_{L^4_T L^\infty} &\lesssim \varepsilon^{\frac{1}{4}}\Big(\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{\dot{B}^{\frac{3}{4}}_{2,1}} + \int_0^T \|(f_\varepsilon, g_\varepsilon)(\tau)\|_{\dot{B}^{\frac{3}{4}}_{2,1}} d\tau\Big)\\ &\lesssim \varepsilon^{\frac{1}{4}}\Big(\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^1} + T\|(f_\varepsilon, g_\varepsilon)\|_{L^\infty_T H^1}\Big).\end{aligned} \tag{36}$$

Where we have used in the last inequality the fact that $H^1 \hookrightarrow B^{\frac{3}{4}}_{2,1} \hookrightarrow \dot{B}^{\frac{3}{4}}_{2,1}$.
To estimate $\|(f_\varepsilon, g_\varepsilon)\|_{H^1}$ we use the following law product

$$\|u \cdot \nabla w\|_{H^1} \lesssim \|u\|_{L^\infty}\|w\|_{H^2} + \|w\|_{L^\infty}\|u\|_{H^2}.$$

Then, by definition of $(f_\varepsilon, g_\varepsilon)$ we have

$$\|(f_\varepsilon, g_\varepsilon)\|_{H^1} \lesssim \|(v_\varepsilon, c_\varepsilon)\|_{L^\infty}\|(v_\varepsilon, c_\varepsilon)\|_{H^2}.$$

Using the embedding $H^2 \hookrightarrow L^\infty$ combined with the energy estimates, we get

$$\begin{aligned}\|(f_\varepsilon, g_\varepsilon)\|_{L^\infty_T H^1} &\lesssim \|(v_\varepsilon, c_\varepsilon)\|^2_{L^\infty_T H^2}\\ &\lesssim \|(v_{0,\varepsilon}, c_{0,\varepsilon})\|^2_{H^2} e^{CV_\varepsilon(T)}.\end{aligned}$$

Inserting this into the estimate (36) we find

$$
\begin{aligned}
\|\Gamma_\varepsilon\|_{L^4_T L^\infty} &\lesssim \varepsilon^{\frac14}\Big(\|(v_{0,\varepsilon},c_{0,\varepsilon})\|_{H^2}+T\|(v_{0,\varepsilon},c_{0,\varepsilon})\|^2_{H^2}e^{CV_\varepsilon(T)}\Big)\\
&\lesssim C_0^\varepsilon\varepsilon^{\frac14}(1+T)e^{CV_\varepsilon(T)}.
\end{aligned}
$$

As the compressible part of v_ε is the imaginary part of Γ_ε, then

$$
\|\mathbb{Q}v_\varepsilon\|_{L^4_T L^\infty} \lesssim C_0^\varepsilon\varepsilon^{\frac14}(1+T)e^{CV_\varepsilon(T)}.
$$

By the same manner, we use (33) to prove

$$
\|c_\varepsilon\|_{L^4_T L^\infty} \lesssim C_0^\varepsilon\varepsilon^{\frac14}(1+T)e^{CV_\varepsilon(T)}.
$$

To prove the second estimate we use an interpolation procedure between the Strichartz estimates for lower frequencies and the energy estimates for higher frequencies. More precisely, consider N an integer that will be judiciously fixed later. Then, using the embedding $B^{s/3}_{\infty,1}\hookrightarrow B^{s/3}_{\infty,\infty}$, Bernstein inequality and the continuity of the operator $\mathbb{Q}$ on the Lebesgue space L^2, we find

$$
\begin{aligned}
\|\mathrm{div}\, v_\varepsilon\|_{B^{s/3}_{\infty,\infty}} &= \|\mathrm{div}\,\mathbb{Q}v_\varepsilon\|_{B^{s/3}_{\infty,\infty}}\\
&\leq \sum_{q<N}2^{q\frac{s}{3}}\|\Delta_q \mathrm{div}\,\mathbb{Q}v_\varepsilon\|_{L^\infty}+\sum_{q\geqslant N}2^{q\frac{s}{3}}\|\Delta_q \mathrm{div}\,\mathbb{Q}v_\varepsilon\|_{L^\infty}\\
&\lesssim \sum_{q<N}2^{q(\frac{s}{3}+1)}\|\Delta_q\mathbb{Q}v_\varepsilon\|_{L^\infty}+2^{-N\frac{s}{3}}\sum_{q\geqslant N}2^{q(\frac{2s}{3}+2)}\|\Delta_q\mathbb{Q}v_\varepsilon\|_{L^2}\\
&\lesssim 2^{N(\frac{s}{3}+1)}\|\mathbb{Q}v_\varepsilon\|_{L^\infty}+2^{-N\frac{s}{3}}\|v_\varepsilon\|_{B^{\frac{2s}{3}+2}_{2,1}}\\
&\lesssim 2^{N(\frac{s}{3}+1)}\|\mathbb{Q}v_\varepsilon\|_{L^\infty}+2^{-N\frac{s}{3}}\|v_\varepsilon\|_{H^{2+s}}.
\end{aligned}
$$

Where we have used in last inequality the embedding $H^{2+s}\hookrightarrow B^{\frac{2s}{3}+2}_{2,1}$. Integrating in time and combining (34) with the energy estimates we get

$$
\begin{aligned}
\|\mathrm{div}\, v_\varepsilon\|_{L^1_T B^{s/3}_{\infty,\infty}} &\lesssim C_0^\varepsilon 2^{N(\frac{s}{3}+1)}\varepsilon^{\frac14}T^{\frac34}(1+T)e^{CV_\varepsilon(t)}+C_0^\varepsilon 2^{-\frac{s}{3}N}Te^{CV_\varepsilon(T)}\\
&\lesssim C_0^\varepsilon(1+T^{\frac74})\big(2^{N(\frac{s}{3}+1)}\varepsilon^{\frac14}+2^{-\frac{s}{3}N}\big)e^{CV_\varepsilon(T)}.
\end{aligned}
$$

By similar computations, we get

$$
\|\nabla c_\varepsilon\|_{L^1_T B^{s/3}_{\infty,\infty}} \lesssim C_0^\varepsilon(1+T^{\frac74})\big(2^{N(\frac{s}{3}+1)}\varepsilon^{\frac14}+2^{-\frac{s}{3}N}\big)e^{CV_\varepsilon(T)}.
$$

We choose N such that $2^{N(\frac{2s}{3}+1)}\approx\varepsilon^{\frac{-1}{4}}$. This is equivalent to take

$$
N\approx\frac{1}{4(\frac{2s}{3}+1)}\log_2\varepsilon^{-1}.
$$

Consequently

$$
\|(\mathrm{div}\ v_\varepsilon,\nabla c_\varepsilon)\|_{L^1_t B^{s/3}_{\infty,\infty}} \lesssim C_0^\varepsilon(1+t^{\frac74})\varepsilon^{\frac{s}{4(2s+3)}}e^{CV_\varepsilon(t)}.
$$

This ends the proof of the corollary.

□

We are now in position to prove our main theorem.

5. Main result

In this section, we shall state more general result than Theorem 1, afterwards, the rest of this section will be devoted to the discussion of the proof. Our result reads as follows.

Theorem 4. *Let $s, \alpha \in]0,1[$, $p \in]1,2[$ and $F \in \mathcal{F}$. Consider a family of initial data $\{(v_{0,\varepsilon}, c_{0,\varepsilon})_{0<\varepsilon<1}\}$ such that there exists a positive constant C which does not depend on ε and verifying*

$$\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^{2+s}} \leq C(\log \varepsilon^{-1})^{\alpha},$$

$$\|\omega_{0,\varepsilon}\|_{L^p \cap BMO_F} \leq C.$$

Then, the system (E.C) admits a unique solution $(v_\varepsilon, c_\varepsilon) \in C\big([0,T_\varepsilon], H^{2+s}\big)$ with the alternative:

1. *If F belongs to the class $\mathcal{F}'$ then the lifespan T_ε of the solution satisfies the lower bound:*

$$T \geqslant \frac{1}{C_0}\mathcal{M}\Big((1-\alpha)\ln\ln\varepsilon^{-1}\Big) \triangleq \tilde{T}_\varepsilon.$$

and the vorticity ω_ε satisfies

$$\forall t \in [0,\tilde{T}_\varepsilon], \quad \|\omega_\varepsilon(t)\|_{BMO_F \cap L^p} \leq C_0(\mathcal{M}^{-1})'\big(C_0(1+t)\big). \tag{37}$$

Where $\mathcal{M} : [0,+\infty[\longrightarrow [0,+\infty[$ is defined by

$$\mathcal{M}(x) \triangleq \int_0^x \frac{dy}{F(e^{Cy})}$$

and $(\mathcal{M}^{-1})'$ denotes the derivative of $\mathcal{M}^{-1}$.

2. *If F belongs to the class $\mathcal{F}\backslash\mathcal{F}'$ then there exists $T_0 > 0$ independent of ε such that for all $t \leq T_0$ we have*

$$\|\omega_\varepsilon(t)\|_{BMO_F \cap L^p} \leq C_0. \tag{38}$$

Moreover, in both cases the compressible and acoustic parts of the solutions tend to zero: there exists $\eta > 0$ such that for small ε and for all $0 \leq T \leq \tilde{T}_\varepsilon$ (respectively $0 \leq T \leq T_0$)

$$\|(\operatorname{div} v_\varepsilon, \nabla c_\varepsilon)\|_{L^1_T B^{s/3}_{\infty,\infty}} \lesssim \varepsilon^{\eta/2}. \tag{39}$$

Assume in addition that $\lim_{\varepsilon\to 0}\|\omega_{0,\varepsilon} - \omega_0\|_{L^p} = 0$, for some vorticity $\omega_0 \in BMO_F \cap L^p$ associated to a divergence-free vector field v_0. Then, the vortices $(\omega_\varepsilon)_\varepsilon$ converge strongly to the solution ω of (1) associated to the initial data ω_0. More precisely, for all $t \in \mathbb{R}_+$ ($0 \leq t \leq T_0$ respectively)

$$\lim_{\varepsilon\to 0}\|\big(\omega_\varepsilon - \omega\big)(t)\|_{L^q} = 0 \quad \forall q \in [p,+\infty[.$$

Furthermore,

(i) *if $F \in \mathcal{F}'$ then for all $t \in \mathbb{R}_+$*

$$\|\omega(t)\|_{BMO_F \cap L^p} \leq C_0(\mathcal{M}^{-1})'\big(C_0(1+t)\big). \tag{40}$$

(ii) *if $F \in \mathcal{F}\backslash\mathcal{F}'$ then for all $t < T_0$*

$$\|\omega(t)\|_{L^p \cap BMO_F} \leq C_0. \tag{41}$$

Remark 5. *Theorem 4 recovers the local and global well-posedness theory of the incompressible Euler system according to the inequalities (40) and (41).*

The proof of Theorem 4 will be divided into two parts: in the first one we estimate the lifespan of the solutions, thereafter, we discuss in the second part the incompressible limit problem.

5.1. **Lower bound of the lifespan.** We will give an a priori bound of T_ε and show that the acoustic parts vanish when the Mach number goes to zero. We denote by

$$W_\varepsilon(t) \triangleq \|v_\varepsilon\|_{L^1_t LL} \quad \text{and} \quad V_\varepsilon(t) \triangleq \|\nabla v_\varepsilon\|_{L^1_t L^\infty} + \|\nabla c_\varepsilon\|_{L^1_t L^\infty}.$$

In view of Theorem 2 and using the embedding (iii) from Proposition 4, we have

$$\|\omega_\varepsilon(t)\|_{BMO_F \cap L^p} \leq C_0 F\Big(e^{CW_\varepsilon(t)}\Big)\Big(1 + F\Big(e^{CW_\varepsilon(t)}\Big)\|\mathrm{div}\, v_\varepsilon\|_{L^1_t B^{\frac{s}{3}}_{\infty,\infty}}\Big) e^{C\|\mathrm{div}\, v_\varepsilon\|_{L^1_t L^\infty}}.$$

According to Remark 2, the function F has at most a polynomial growth : $F(x) \lesssim 1 + x^\beta$. Also, since $\|v_\varepsilon\|_{LL} \lesssim \|\nabla v_\varepsilon\|_{L^\infty}$ we may write

$$\|\omega_\varepsilon(t)\|_{BMO_F \cap L^p} \quad \leq \quad C_0 F\Big(e^{CW_\varepsilon(t)}\Big)\Big(1 + e^{CV_\varepsilon(t)}\|\mathrm{div}\, v_\varepsilon\|_{L^1_t B^{\frac{s}{3}}_{\infty,\infty}}\Big) e^{C\|\mathrm{div}\, v_\varepsilon\|_{L^1_t L^\infty}}.$$

It follows from Corollary 1 that

$$\begin{aligned}\|\omega_\varepsilon(t)\|_{BMO_F \cap L^p} &\leq C_0 F\Big(e^{C\|v_\varepsilon\|_{L^1_t LL}}\Big)\Big(1 + C_0^\varepsilon(1+t^{\frac{7}{4}})\varepsilon^\eta e^{CV_\varepsilon(t)}\Big) e^{C_0^\varepsilon(1+t^{\frac{7}{4}})\varepsilon^\eta e^{CV_\varepsilon(t)}} \\ &\leq C_0 F\Big(e^{CW_\varepsilon(t)}\Big) e^{C_0^\varepsilon(1+t^{\frac{7}{4}})\varepsilon^\eta e^{CV_\varepsilon(t)}}, \end{aligned} \tag{42}$$

we recall that C_0^ε is a positive constant depending polynomially on $\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^{2+s}}$.
Now Lemma 2 and the embeddings $B^s_{\infty,\infty} \hookrightarrow BMO \hookrightarrow B^0_{\infty,\infty}$ ensure that

$$\|v_\varepsilon\|_{LL} \quad \lesssim \quad \|\omega_\varepsilon\|_{BMO_F \cap L^p} + \|\mathbb{Q} v_\varepsilon\|_{L^\infty} + \|\mathrm{div}\, v_\varepsilon\|_{B^s_{\infty,\infty}}. \tag{43}$$

Integrating in time and using Corollary 1 we get

$$W_\varepsilon(t) \quad \leq \quad C\int_0^t \|\omega_\varepsilon(\tau)\|_{BMO_F \cap L^p} d\tau + C_0^\varepsilon(1+t^{\frac{7}{4}})\varepsilon^\eta e^{CV_\varepsilon(t)}. \tag{44}$$

Setting

$$\rho_\varepsilon(t) \triangleq C_0^\varepsilon(1+t^{\frac{7}{4}})\varepsilon^\eta e^{CV_\varepsilon(t)}$$

and inserting the estimate (42) into (44) give

$$W_\varepsilon(t) \leq C_0 \int_0^t F\big(e^{CW_\varepsilon(\tau)}\big) e^{\rho_\varepsilon(\tau)} d\tau + \rho_\varepsilon(t). \tag{45}$$

At this stage we distinguish two cases depending whether $F \in \mathcal{F}'$ or not.

(1) If $F \in \mathcal{F}'$, we fix $T > 0$ an arbitrary real number. So the inequality (45) becomes

$$\forall t \in [0,T] \quad W_\varepsilon(t) \quad \leq \quad C_0 \int_0^t F\big(e^{CW_\varepsilon(\tau)}\big) e^{\rho_\varepsilon(\tau)} d\tau + \rho_\varepsilon(T).$$

We introduce the function $\mathcal{M} : [0,+\infty[\longrightarrow [0,+\infty[$ defined by

$$\mathcal{M}(x) \triangleq \int_0^x \frac{dy}{F(e^{Cy})}.$$

Since $\mathcal{M}$ is a nondecreasing function and $\lim_{x\to\infty} \mathcal{M}(x) = +\infty$ then $\mathcal{M}$ is one-to one and Lemma 3 implies that

$$\forall t \in [0,T] \quad W_\varepsilon(t) \quad \leq \quad \mathcal{M}^{-1}\Big(\mathcal{M}\big(\rho_\varepsilon(T)\big) + C_0 e^{\rho_\varepsilon(t)} t\Big).$$

Then,

$$W_\varepsilon(T) \quad \leq \quad \mathcal{M}^{-1}\Big(\mathcal{M}\big(\rho_\varepsilon(T)\big) + C_0 e^{\rho_\varepsilon(T)} T\Big).$$

Inserting this estimate into (42) leads to

$$\|\omega_\varepsilon(T)\|_{BMO_F \cap L^p} \quad \leq \quad C_0 F\Big(e^{C\mathcal{M}^{-1}\big(\mathcal{M}(\rho_\varepsilon(T)) + C_0 e^{\rho_\varepsilon(T)} T\big)}\Big) e^{\rho_\varepsilon(T)}. \tag{46}$$

Now we need the following estimate whose proof is given in the Appendix:

$$\|\nabla v_\varepsilon(T)\|_{L^\infty} \lesssim \|(v_{0,\varepsilon}, c_{0\varepsilon})\|_{H^{s+2}} + \|\omega_\varepsilon(T)\|_{BMO\cap L^p} V_\varepsilon(T).$$

This, combined with Corollary 1 yield

$$V_\varepsilon(T) \le \rho_\varepsilon(T) + C_0^\varepsilon T + C\int_0^T \|\omega_\varepsilon(t)\|_{BMO\cap L^p} V_\varepsilon(t)dt.$$

Hence Gronwall's inequality implies that

$$V_\varepsilon(T) \le \big(C_0^\varepsilon T + \rho_\varepsilon(T)\big)\exp\Big(C\int_0^T \|\omega_\varepsilon(t)\|_{BMO\cap L^p}dt\Big). \tag{47}$$

Putting (46) in the last estimate we get

$$V_\varepsilon(T) \quad \le \quad \big(C_0^\varepsilon T + \rho_\varepsilon(T)\big)\exp\Big(C_0 e^{\rho_\varepsilon(T)}\int_0^T F\big(e^{C\mathcal{M}^{-1}(\mathcal{M}(\rho_\varepsilon(t))+C_0e^{\rho_\varepsilon(t)}t)}\big)dt\Big).$$

Assuming $\rho_\varepsilon(T) \le 1$, which is true at least for small T, and using the fact that $\mathcal{M}^{-1}$ is non decreasing we find

$$V_\varepsilon(T) \lesssim C_0^\varepsilon(1+T)\exp\Big(C_0\int_0^T F\big(e^{C\mathcal{M}^{-1}(C_0(1+t))}\big)dt\Big).$$

Moreover the inequality (46) becomes

$$\|\omega_\varepsilon(T)\|_{BMO_F\cap L^p} \quad \le \quad C_0 F\Big(e^{C\mathcal{M}^{-1}\big(C_0(1+T)\big)}\Big).$$

A straightforward computation gives

$$F\big(e^{C\mathcal{M}^{-1}(x)}\big) = (\mathcal{M}^{-1})'(x).$$

Then we immediately deduce that

$$\|\omega_\varepsilon(T)\|_{BMO_F\cap L^p} \quad \le \quad C_0(\mathcal{M}^{-1})'\big(C_0(1+T)\big),$$

and

$$V_\varepsilon(T) \lesssim C_0^\varepsilon(1+T)\exp\Big(\mathcal{M}^{-1}\big(C_0(1+T)\big)\Big)$$

So from the condition $\rho_\varepsilon(T) \le 1$ we can conclude a lower bound of the lifespans of the solution. Indeed, we have

$$\begin{aligned}\rho_\varepsilon(T) \quad &= \quad C_0^\varepsilon\varepsilon^\eta(1+T^{\frac{7}{4}})e^{CV_\varepsilon(T)}\\ &\le \quad \varepsilon^\eta\exp\Big(C_0^\varepsilon(1+T)e^{\mathcal{M}^{-1}\big(C_0(1+T)\big)}\Big).\end{aligned}$$

Now let $\alpha(\varepsilon)$ be a function going to ∞ as ε approaches zero and choosing T such that

$$C_0(1+T) = \mathcal{M}(\alpha(\varepsilon))$$

Then in order to get $\rho_\varepsilon(T) \le \varepsilon^{\frac{\eta}{2}}$ we should impose the constraint

$$\exp\Big(C_0^\varepsilon\mathcal{M}(\alpha(\varepsilon))e^{\alpha(\varepsilon)}\Big) \le \varepsilon^{-\frac{\eta}{2}}.$$

Since $F(x) \ge 1$, for $x \ge 1$ then from the definition of $\mathcal{M}$ we infer that

$$\mathcal{M}(x) \le x$$

and thus to get the preceding inequality it suffices to assume

$$\exp\Big(C_0^\varepsilon e^{\alpha(\varepsilon)}\Big) \le \varepsilon^{-\frac{\eta}{2}}.$$

At this stage we see that one can impose the following conditions,

$$C_0^\varepsilon \le C(\ln\varepsilon^{-1})^\alpha \quad \text{and} \quad \alpha(\varepsilon) \approx (1-\alpha)\ln\ln\frac{1}{\varepsilon}$$

for some $\alpha \in (0,1)$. In particular, from Corollary 1, we conclude that

$$\|(\text{div}v_\varepsilon, \nabla c_\varepsilon)\|_{(L^1_T\cap L^4_T)B^{\frac{s}{3}}_{\infty,\infty}} + \|(\mathbb{Q}v_\varepsilon, c_\varepsilon)\|_{(L^1_T\cap L^4_T)L^\infty} \lesssim \varepsilon^{\eta/2} \tag{48}$$

(2) Let $F \in \mathcal{F}\backslash\mathcal{F}'$ then we return to the estimate (45) and we assume that $\rho_\varepsilon(t) \leq 1$. Using again the fact that F has at most a polynomial growth, we get

$$W_\varepsilon(t) \leq C_0 e^{CW_\varepsilon(t)} t + 1.$$

Consequently we can find $T_0 \in (0,1)$ independent of ε such that

$$\forall t \in [0, T_0] \quad W_\varepsilon(t) \leq 2.$$

Plugging this estimate into (42) leads to

$$\|\omega_\varepsilon(t)\|_{BMO_F \cap L^p} \leq C_0.$$

This combined with (47) and the constraint on ρ_ε give

$$\begin{aligned} V_\varepsilon(T) &\leq C_0^\varepsilon (T+1) e^{C_0 T} \\ &\lesssim C_0^\varepsilon. \end{aligned}$$

Hence,

$$\begin{aligned} \rho_\varepsilon(T) &= C_0^\varepsilon \varepsilon^\eta (1 + T^{\frac{7}{4}}) e^{CV_\varepsilon(T)} \\ &\lesssim \varepsilon^\eta e^{2C_0^\varepsilon}. \end{aligned}$$

Choosing $C_0^\varepsilon \leq C(\ln \varepsilon^{-1})^\alpha$, the last expression will be bounded by $\varepsilon^{\frac{\eta}{2}}$ and thus we find $\rho_\varepsilon(t) \leq 1$. This concludes the proof of the first part of Theorem 4.

5.2. **Incompressible limit.**

Proof. As it has already pointed out, the vorticity ω_ε has the following structure

$$\omega_\varepsilon(t) = \omega_{0,\varepsilon}(\psi_\varepsilon^{-1}(t)) e^{-\int_0^t \text{div}v_\varepsilon(\tau, \psi(\tau, \psi_\varepsilon^{-1}(t)))d\tau}. \tag{49}$$

So the question of the convergence of the vortices $\{\omega_\varepsilon\}$ can be examined through the convergence of the flow maps $\{\psi_\varepsilon\}$. In other words, we shall establish the existence of the particle trajectories ψ as a uniform limit of a subsequence of $\{\psi_\varepsilon\}$. Once this flow is constructed, we can propose a candidate for the solution of the incompressible Euler system given by

$$\omega(t,x) = \omega_0(\psi^{-1}(t,x)) \quad \text{and} \quad v(t) = K \star \omega(t), \quad K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}. \tag{50}$$

At this stage and in order to get a solution for (E.I) we need to show that ψ is the flow map associated to the velocity v. For this goal we develop strong convergence results of the velocities $\{v_\varepsilon\}$.
To begin with, let T and R be two positive real numbers such tha $T \leq \tilde{T}_\varepsilon$. Then, for all $t \in [0,T]$ and $x \in \bar{B}(0,R)$ we use the integral equation of the flow ψ_ε to get

$$\begin{aligned} |\psi_\varepsilon^{-1}(t,x) - x| &= \left| \int_0^t v_\varepsilon(\tau, \psi_\varepsilon(\tau, \psi_\varepsilon^{-1}(t,x))) d\tau \right| \\ &\leq \int_0^t \|v_\varepsilon(\tau)\|_{L^\infty} d\tau \\ &\leq \|\mathbb{P}v_\varepsilon\|_{L^1_T L^\infty} + \|\mathbb{Q}v_\varepsilon(\tau)\|_{L^1_T L^\infty}. \end{aligned} \tag{51}$$

Since the incompressible part $\mathbb{P}v_\varepsilon$ has the same vorticity as the total velocity

$$\text{curl } \mathbb{P}v_\varepsilon = \text{curl } v_\varepsilon,$$

and $1 < p < 2$, the Biot-Savart law implies that

$$\|\mathbb{P}v_\varepsilon(\tau)\|_{L^\infty} \lesssim \|\omega_\varepsilon(\tau)\|_{L^p \cap L^{2p}}. \tag{52}$$

Hence, according to the identity (49) we find

$$\|\mathbb{P}v_\varepsilon(\tau)\|_{L^\infty} \lesssim \|\omega_{0,\varepsilon}(\psi_\varepsilon^{-1}(\tau))\|_{L^p\cap L^{2p}} e^{\|\operatorname{div} v_\varepsilon\|_{L^1_T L^\infty}}.$$

Using a change of variable combined with Lemma 6 and the estimate (48) we get

$$\begin{aligned}\|\mathbb{P}v_\varepsilon(\tau)\|_{L^\infty} &\lesssim \|\omega_{0,\varepsilon}\|_{L^p\cap L^{2p}} e^{C\|\operatorname{div} v_\varepsilon\|_{L^1_T L^\infty}}\\ &\lesssim \|\omega_{0,\varepsilon}\|_{L^p\cap BMO} e^{C\varepsilon^{\eta/2}}\\ &\leq C_0, \end{aligned}\tag{53}$$

where we have used the classical interpolation inequality (17). Plugging the estimates (48) and (53) into (51) we find

$$|\psi_\varepsilon^{-1}(t,x)| \leq C_0T+R+1. \tag{54}$$

So the family $\{\psi_\varepsilon^{-1}\}$ is uniformly bounded on every compact $[0,T]\times\bar{B}(0,R)$ and it remains to study its equicontinuity. According to the Lemma 4 we have

$$\forall (x,y)\in\bar{B}(0,R)^2,\quad |x-y|\leq e^{-\exp\left(\|v_\varepsilon\|_{L^1_T LL}\right)} \Rightarrow |\psi_\epsilon^{-1}(t,x)-\psi_\epsilon^{-1}(t,y)|\leq e|x-y|^{\exp(-\|v_\varepsilon\|_{L^1_T LL})}.$$

But estimate (43) combined with (37) and (48) ensures the existence of an explicit time continuous function $\alpha(t)>0$ such that

$$\|v_\varepsilon\|_{L^1_T LL}\leq C_0\alpha(T).$$

Hence, for all $(x,y)\in\bar{B}(0,R)^2$ with $|x-y|\leq e^{-\exp(C_0\alpha(T))}$ we have

$$\left|\psi_\epsilon^{-1}(t,x)-\psi_\epsilon^{-1}(t,y)\right|\leq e|x-y|^{\exp(-C_0\alpha(T))}. \tag{55}$$

Consider the backward particle trajectories that we denote by $(\phi_\varepsilon(s,t,x))_\varepsilon$ and which satisfies,

$$\phi_\varepsilon(s,t,x)=x-\int_s^t v_\varepsilon(\tau,\phi_\varepsilon(\tau,t,x))d\tau.$$

Then it is a well-known fact that

$$\phi_\varepsilon(0,t,x)=\psi^{-1}(t,x)\quad\text{and}\quad \phi_\varepsilon(0,t_2,x)=\phi_\varepsilon\big(0,t_1,\phi_\varepsilon(t_1,t_2,x)\big)\quad\forall (t_1,t_2)\in[0,T]^2.$$

Consequently we get in view of (55)

$$\begin{aligned}\left|\psi_\varepsilon^{-1}(t_1,x)-\psi_\varepsilon^{-1}(t_2,x)\right| &= \left|\psi_\varepsilon^{-1}(t_1,x)-\psi_\varepsilon^{-1}(t_1,\phi_\varepsilon(t_1,t_2,x))\right|\\ &\leq e\big|x-\phi_\varepsilon(t_1,t_2,x)\big|^{\exp(-C_0\alpha(T))}\\ &= e\Big|\int_{t_1}^{t_2} v_\varepsilon(\tau,\phi_\varepsilon(\tau,t_2,x))d\tau\Big|^{\exp(-C_0\alpha(T))}\\ &\leq e\Big|\int_{t_1}^{t_2}\big(\|\mathbb{P}v_\varepsilon(\tau)\|_{L^\infty}+\|\mathbb{Q}v_\varepsilon(\tau)\|_{L^\infty}\big)d\tau\Big|^{\exp(-C_0\alpha(T))}.\end{aligned}$$

despite that

$$\big|x-\phi_\varepsilon(t_1,t_2,x)\big|\leq e^{-\exp(C_0\alpha(T))}. \tag{56}$$

It follows from (48) and (53) that

$$\left|\psi_\varepsilon^{-1}(t_1,x)-\psi_\varepsilon^{-1}(t_2,x)\right|\leq e\Big|C_0|t_1-t_2|+C\varepsilon^{\eta/2}|t_1-t_2|^{3/4}\Big|^{\exp(-C_0\alpha(T))}.$$

By taking $|t_2-t_2|<<1$ we see that the condition (56) is satisfied and the preceding estimate is justified. Thus with (55) we find that the family $\{\psi_\epsilon^{-1}\}$ is equicontinuous on every compact $[0,T]\times\bar{B}(0,R)$. Consequently, the Arzela-Ascoli theorem ensures the existence of a subsequence, denoted also by $\{\psi_\epsilon^{-1}\}$ and a particle trajectories ψ^{-1}, such that $\{\psi_\varepsilon^{-1}\}$ converges uniformly to ψ^{-1} on every compact $[0,T]\times\bar{B}(0,R)$. Observe that the subsequence may in principle depend on this compact but we can suppress this dependence by using Cantor's diagonal argument.

Performing the same analysis as previously to the integral equation of the flow ϕ_ε we can readily obtain that up to an extraction $\{\phi_\varepsilon\}$ converges uniformly in any compact to some continuous function ϕ. Moreover for any $t,s\in[0,T]$, $\phi(t,s)$ is a homeomorphism with

$$\phi^{-1}(t,s,x)=\phi(s,t,x),\quad \psi^{-1}(t,x)=\phi(0,t,x),\quad \psi(t,x)=\phi(t,0,x).$$

In addition, for all $t\in[0,T]$, $\psi(t)$ is a Lebesgue measure preserving map. More precisely, for $q\in[1,\infty[$ and $f\in L^q(\mathbb{R}^2)$ we have

$$\|f\circ\psi_t^{\pm1}\|_{L^q}=\|f(t)\|_{L^q}. \tag{57}$$

Indeed, for all continuous, compactly supported function f, $\{f\circ\psi_{t,\varepsilon}^{\pm1}\}$ converges pointwisely to $f\circ\psi_t^{\pm1}$. By the uniform boundedness of $\{\psi_{t,\varepsilon}^{\pm1}\}$ with respect to ε, we get from the integral equation,

$$|x|\le|\psi_\varepsilon^{\pm1}(t,x)|+C_0T\quad\forall t\in[0,T].$$

Since f is compactly supported, we conclude the existence of $M>0$ such that

$$\text{supp}(f\circ\psi_{t,\varepsilon})\subset B(0,M+C_0T).$$

Therefore, by Lebesgue dominated convergence theorem, we get

$$\lim_{\varepsilon\to0}\|f\circ\psi_{t,\varepsilon}^{\pm1}-f\circ\psi_t^{\pm1}\|_{L^q}=0. \tag{58}$$

In particular,

$$\lim_{\varepsilon\to0}\|f\circ\psi_\varepsilon^{\pm1}\|_{L^q}=\|f\circ\psi^{\pm1}\|_{L^q}.$$

On other hand, a change of variable combined with Lemma 6 lead to

$$\|f\|_{L^q}e^{-C\int_0^t\|\text{div}v_\varepsilon(\tau)\|_{L^\infty}d\tau}\lesssim\|f\circ\psi_\varepsilon^{\pm1}\|_{L^q}\lesssim\|f\|_{L^q}e^{C\int_0^t\|\text{div}v_\varepsilon(\tau)\|_{L^\infty}d\tau}.$$

Taking into consideration the estimate (48), the passage to the limit in the last inequality gives the identity (57). To finish the proof we use a density argument.
With this flow ψ in hand we construct (v,ω) via (50) and we shall prove some strong convergence results which give in turn that (v,ω) is a solution of the incompressible Euler equations. Recall that ω_0 and $(\omega_\varepsilon)_\varepsilon$ belong to L^q for all $q\in[p,+\infty[$ according to the classical interpolation result between Lebesgue and BMO spaces, see (17). Then, we shall prove following convergence result,

$$\lim_{\varepsilon\to0}\|(\omega_\varepsilon-\omega)(t)\|_{L^q}\quad\forall q\in[p,+\infty[,\quad\forall t\in[0,T].$$

For this aim we write

$$\begin{aligned}\|(\omega-\omega_\varepsilon)(t)\|_{L^q}&=\|\omega_0\circ\psi^{-1}(t)-\omega_{0,\varepsilon}\circ\psi_\varepsilon^{-1}(t)e^{-\int_0^t\text{div}v_\varepsilon(\tau,\psi(\tau,\psi_\varepsilon^{-1}(t)))d\tau}\|_{L^q}\\&\le\text{I}_\varepsilon+\text{II}_\varepsilon.\end{aligned}$$

Where

$$\text{I}_\varepsilon\triangleq\|\omega_0\circ\psi^{-1}(t)-\omega_0\circ\psi_\varepsilon^{-1}(t)\|_{L^q},$$

and

$$\text{II}_\varepsilon\triangleq\|\omega_0\circ\psi_\epsilon^{-1}(t)-\omega_0^\epsilon\circ\psi_\varepsilon^{-1}(t)e^{-\int_0^t\text{div}v_\varepsilon(\tau,\psi(\tau,\psi_\varepsilon^{-1}(t)))d\tau}\|_{L^q}.$$

In view of the equality (58), we can confirm that

$$\lim_{\varepsilon\to0}\text{I}_\varepsilon=0.$$

To estimate II_ε we make a change of variable and we use Lemma 6 to get

$$\begin{aligned}\text{II}_\varepsilon&\le e^{C\|\text{div}v_\varepsilon\|_{L_t^1L^\infty}}\|\omega_0-\omega_{0,\varepsilon}e^{-\int_0^t\text{div}v_\varepsilon(\tau,\psi(\tau))d\tau}\|_{L^q}\\&\le e^{C\|\text{div}v_\varepsilon\|_{L_t^1L^\infty}}\Big(\|\omega_0\big(1-e^{-\int_0^t\text{div}v_\varepsilon(\tau,\psi(\tau))d\tau}\big)\|_{L^q}+\|\big(\omega_0-\omega_{0,\varepsilon}\big)e^{-\int_0^t\text{div}v_\varepsilon(\tau,\psi(\tau))d\tau}\|_{L^q}\Big)\\&\le e^{C\|\text{div}v_\varepsilon\|_{L_t^1L^\infty}}\Big(\|\omega_0\|_{L^q}\|1-e^{-\int_0^t\text{div}v_\varepsilon(\tau,\psi(\tau))d\tau}\|_{L^\infty}+\|\omega_0-\omega_{0,\varepsilon}\|_{L^q}\Big)\\&\le e^{C\|\text{div}v_\varepsilon\|_{L_t^1L^\infty}}\Big(\|\omega_0\|_{L^q}\int_0^t\|\text{div}v_\varepsilon(\tau)\|_{L^\infty}d\tau+\|\omega_0-\omega_{0,\varepsilon}\|_{L^q}\Big).\end{aligned}$$

Where we have used in the last inequality the estimate

$$\|e^u - 1\|_{L^\infty} \le \|u\|_{L^\infty} e^{\|u\|_{L^\infty}}.$$

Then, from (39) and (17) we find

$$\begin{aligned}
\mathrm{II}_\varepsilon &\lesssim C_0\varepsilon^\eta + \|\omega_0 - \omega_{0,\varepsilon}\|_{L^q}\\
&\lesssim C_0\varepsilon^\eta + \|\omega_0 - \omega_{0,\varepsilon}\|_{L^p}^{\frac{p}{q}} \|\omega_0 - \omega_{0,\varepsilon}\|_{BMO}^{1-\frac{p}{q}}\\
&\lesssim C_0\varepsilon^\eta + C_0\|\omega_0 - \omega_{0,\varepsilon}\|_{L^p}^{\frac{p}{q}}.
\end{aligned}$$

Passing to the limit in the last estimate gives the desired result. Now we shall translate these results to the velocities via Biot-Savart law: we get since $1 < p < 2$,

$$\|\big(\mathbb{P}v_\varepsilon - v\big)(t)\|_{L^\infty} \lesssim \|\big(\omega_\varepsilon - \omega\big)(t)\|_{L^p\cap L^{2p}}. \tag{59}$$

Furthermore, by the classical Hardy-Littlewood-Sobolev inequality, one has

$$\|\big(\mathbb{P}v_\varepsilon - v\big)(t)\|_{L^r} \lesssim \|\big(\omega_\varepsilon - \omega\big)(t)\|_{L^q}. \tag{60}$$

where $r \in [\frac{2p}{2-p}, +\infty[$ and $q \in [p, +\infty[$. Moreover, in view of the Calderón-Zygmund inequality (8) we have

$$\|\nabla\big(\mathbb{P}v_\varepsilon - v\big)(t)\|_{L^q} \lesssim \|\big(\omega_\varepsilon - \omega\big)(t)\|_{L^q} \quad \forall q \in [p, +\infty[.$$

Then, the convergence of $(\mathbb{P}v_\varepsilon)$ to v holds true in $W^{1,r}$ for all $r \in [\frac{2p}{2-p}, +\infty[$.

It remains to show that ω is a solution of (1) associated to the initial vorticity ω_0. But before doing it, we have to verify that ψ is the flow associated to v. Using the preceding convergence and the uniform convergence of $\{\psi_\varepsilon\}$ and according to the estimate (48), the passage to the limit in the integral equation of the flow,

$$\psi_\varepsilon(t,x) = x + \int_0^t \mathbb{P}v_\varepsilon(\tau, \psi_\varepsilon(\tau,x))d\tau + \int_0^t \mathbb{Q}v_\varepsilon(\tau, \psi_\varepsilon(\tau,x))d\tau,$$

yields

$$\psi(t,x) = x + \int_0^t v(\tau, \psi(\tau,x))d\tau.$$

As $v \in LL$ then by the uniqueness of the flow associated to v we can confirm the assumption.
Next, let ϕ be an element of $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^2)$. By definition of ω and using a change of variable we have

$$\int_0^\infty \int_{\mathbb{R}^2} \omega(t,x)\partial_t\phi(t,x)dxdt = \int_0^\infty \int_{\mathbb{R}^2} \omega_0(x)(\partial_t\phi)(t,\psi(t,x))dxdt.$$

But

$$\begin{aligned}
(\partial_t\phi)(t,\psi(t,x)) &= \partial_t\big(\phi(t,\psi(t,x))\big) - \partial_t\psi(t,x)\cdot\nabla\phi(t,\psi(t,x))\\
&= \partial_t\big(\phi(t,\psi(t,x))\big) - (v\cdot\nabla\phi)(t,\psi(t,x)).
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^2} \omega(t,x)\partial_t\phi(t,x)dxdt &= -\int_{\mathbb{R}^2} \omega_0(x)\phi(0,x)dx - \int_0^\infty \int_{\mathbb{R}^2} \omega_0(x)(v\cdot\nabla\phi)(t,\psi(t,x))dxdt\\
&= -\int_{\mathbb{R}^2} \omega_0(x)\phi(0,x)dx - \int_0^\infty \int_{\mathbb{R}^2} \omega(t,x)(v\cdot\nabla\phi)(t,x)dxdt.
\end{aligned}$$

Hence, ω verifies the velocity-vorticity weak formulation :

$$\int_0^\infty \int_{\mathbb{R}^2} \omega(t,x)(\partial_t\phi + v\cdot\nabla\phi)(t,x)dxdt + \int_{\mathbb{R}^2} \omega_0(x)\phi(0,x)dx = 0.$$

Moreover, from (iv) of Proposition 1 and the estimates (37), (38) we immediately deduce (40) and (41). Finally, the uniqueness of the limit can be concluded by uniqueness of the solution of the incompressible Euler system since the velocity v belongs to $L^1_T LL$. $\square$

6. Appendix

Lemma 8. *Let B be a ball of center 0 and radius $r>0$ and ψ be the flow associated to a given smooth vector field v. Consider a Whitney covering of the open connected set $\psi(t,B)$ that is a collection of countable open balls $(O_k)_k$ introduced in the proof of Theorem 3.*
For all $k\in\mathbb{N}$ we set

$$U_k \triangleq \sum_{e^{-k-1}h(r)<r_j\le e^{-k}h(r)} |O_j|,$$

and

$$V_k \triangleq \sum_{e^{-k-1}<4r_j\le e^{-k}} |O_j|,$$

with $h(r)\triangleq r\max\{1,\|J_\psi\|_{L^\infty}\}$ and J_ψ is the Jacobian of ψ.
Then, there exists an absolute constants C such that for all $k\ge\beta(t)$, we have
If $r\lesssim e^{-\beta(t)}\min\big\{1,\frac{1}{\|J_\psi\|_{L^\infty}}\big\}$, then

$$U_k \le C\big(1+\|J_\psi\|_{L^\infty}\big)^2 e^{-\frac{k}{\beta(t)}} r^{1+\frac{1}{\beta(t)}}, \tag{61}$$

For all $r\in\mathbb{R}_+^*$,

$$V_k \le C\|J_\psi\|_{L^\infty} e^{-\frac{k}{\beta(t)}} r. \tag{62}$$

Where $\beta(t)=\exp\Big(\int_0^t\|v(\tau)\|_{LL}d\tau\Big)$.

Proof By the definition of U_k, we have

$$U_k \le \Big|\Big\{y\in\psi(B): d(y,\psi(B)^c)\le Ce^{-k}r\max\{1,\|J_\psi\|_{L^\infty}\}\Big\}\Big|$$

Since $|\psi(A)|\le|A|\|J_\psi\|_{L^\infty}$ for any measurable set $A\subset\mathbb{R}^2$, we can deduce that

$$U_k \le \Big|\Big\{x\in B: d(\psi(x),\psi(B^c))\le Ce^{-k}r\max\{1,\|J_\psi\|_{L^\infty}\}\Big\}\Big|\|J_\psi\|_{L^\infty}. \tag{63}$$

We set

$$D_k=\Big\{x\in B: d(\psi(x),\psi(B^c))\le Ce^{-k}r\max\{1,\|J_\psi\|_{L^\infty}\}\Big\}.$$

According to the fact $d(\psi(x),\psi(B^c))=d(\psi(x),\partial\psi(B))$ and $\partial\psi(B)=\psi(\partial B)$, we can write

$$D_k\subset\Big\{x\in B:\exists y\in\partial B: |\psi(x)-\psi(y)|\le Ce^{-k}r\max\{1,\|J_\psi\|_{L^\infty}\}\Big\}.$$

As $r\lesssim e^{-\beta(t)}\min\big\{1,\frac{1}{\|J_\psi\|_{L^\infty}}\big\}$, then

$$e^{-k}r\max\{1,\|J_\psi\|_{L^\infty}\}\lesssim e^{-\beta(t)}.$$

Then, Lemma 4 applied with ψ^{-1} gives

$$D_k \quad\subset\quad \Big\{x\in B:\exists y\in\partial B: |x-y|\le Ce^{1-\frac{k}{\beta(t)}}r^{\frac{1}{\beta(t)}}(1+\|J_\psi\|_{L^\infty})\Big\}.$$

Therefore,

$$D_k\subset A=\Big\{x\in B: d(x,\partial B): |x-y|\le Ce^{1-\frac{k}{\beta(t)}}r^{\frac{1}{\beta(t)}}(1+\|J_\psi\|_{L^\infty})\Big\}.$$

Inserting this into (63) gives

$$U_k\le\|J_\psi\|_{L^\infty}|D_k|\lesssim\big(1+\|J_\psi\|_{L^\infty}\big)^2 e^{\frac{-k}{\beta(t)}}r^{1+\frac{1}{\beta(t)}}$$

as claimed. Reproducing the same procedure as previously with replacing $Ce^{-k}r\max\{1,\|J_\psi\|_{L^\infty}\}$ by Ce^{-k} in (63) and considering the fact that $c_0e^{-k}\lesssim e^{-\beta(t)}$, we get the estimation of V_k.

Lemma 9. *Let $(v_\varepsilon, c_\varepsilon)$ be a smooth solution of the compressible Euler system (E.C) and ω_ε be the vorticity of v_ε. Then there exists a positive constant C such that*

$$\|\nabla v_\varepsilon(t)\|_{L^\infty} \le C\Big(\|\omega_\varepsilon(t)\|_{BMO\cap L^p} V_\varepsilon(t) + \|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^{s+2}}\Big),$$

with

$$V_\varepsilon(t) = \int_0^t \big(\|\nabla v_\varepsilon(\tau)\|_{L^\infty} + \|\nabla c_\varepsilon(\tau)\|_{L^\infty}\big)d\tau$$

Proof. According to Bernstein ineguality and the fact that $\|\dot{\Delta v_{\varepsilon}}_q\|_{L^\infty} \sim 2^{-q}\|\dot\Delta_q \omega\|_{L^\infty}$, we have

$$\begin{aligned}
\|\nabla v_\varepsilon\|_{L^\infty} &\le \|\Delta_{-1}\nabla v_\varepsilon\|_{L^\infty} + \sum_{0\le q\le N} \|\Delta_q \nabla v_\varepsilon\|_{L^\infty} + \sum_{q\ge N}\|\Delta_q \nabla v_\varepsilon\|_{L^\infty}\\
&\lesssim \|\Delta_{-1}\nabla v_\varepsilon\|_{L^p} + \sum_{0\le q\le N}\|\Delta_q \omega_\varepsilon\|_{L^\infty} + \sum_{q\ge N} 2^q\|\Delta_q v_\varepsilon\|_{L^\infty}\\
&\lesssim \|\omega_\varepsilon\|_{L^p} + N\|\omega_\varepsilon\|_{B^0_{\infty,\infty}} + \|v_\varepsilon\|_{B^{s+1}_{\infty,\infty}} \sum_{q\ge N} 2^{-qs}\\
&\lesssim N\|\omega_\varepsilon\|_{BMO_F\cap L^p} + 2^{-Ns}\|v_\varepsilon\|_{H^{s+2}}.
\end{aligned}$$

where we have used in the last inequality the fact that $H^{s+2} \hookrightarrow B^{s+1}_{\infty,\infty}$. Then from the energy estimates we deduce that

$$\|\nabla v_\varepsilon(t)\|_{L^\infty} \lesssim N\|\omega_\varepsilon(t)\|_{BMO_F\cap L^p} + 2^{-Ns}\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{H^{s+2}} e^{CV_\varepsilon(t)}.$$

Choosing N such that $2^{Ns} \simeq e^{CV_\varepsilon(t)}$, gives the desired result.

□

References

[1] T. Alazard. *Incompressible limit of the nonisentropic Euler equations with the solid wall boundary conditions.* Adv. Differential Equations 10 (2005), no. 1, 19–44.

[2] S. Alinhac. *Temps de vie des solutions régulières des équations d'Euler compressibles axisymétriques en dimension deux.* Invent. Math. **111** (1993), no. **3**, 627–670.

[3] K. Asano, *On the incompressible limit of the compressible Euler equation*, Japan J. Appl. Math., 4 (1987), no. 3, 455–488.

[4] H. Bahouri, J-Y. Chemin and R. Danchin,*Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften 343.

[5] F. Bernicot and T. Hmidi, *On the global well-posedness for Euler equations with unbounded vorticity*, preprint arXiv:1303.6151 (2013).

[6] F. Bernicot and S. Keraani, *On the global well-posedness of the 2D Euler equations for a large class of Yudovich type data*, preprint arXiv:1204.6006 (2012).

[7] J.-Y. Chemin, *Fluides Parfaits Incompressibles*, , Astérisque 230 (1995); *Perfect Incompressible Fluids* transl.by I. Gallagher and D. Iftimie, Oxford Lecture Series in Mathematics and Its Applications, Vol. 14, Clarendon Press-Oxford University Press, New York (1998).

[8] J.-M. Delort, *Existence de nappes de tourbillon en dimension deux*, J. Amer. Math. Sot., Vol. 4 (1991) 553-586.

[9] A. Dutrifoy and T. Hmidi, *The incompressible limit of solutions of the two-dimensional compressible Euler system with degenerating initial data* Comm. Pure Appl. Math. 57 (2004), no. 9, 1159–1177.

[10] I. Gallagher. *Résultats récents sur la limite incompressible.* Séminaire Bourbaki. Vol. 2003/2004. Astérisque No. 299 (2005), Exp. No. 926, *vii* , 29 – 57.

[11] R. DiPerna and A. Madja, *Concentrations in regularization for 2D incompressible flow.* Comm. Pure Appl. Math. 40 (1987), 301-345.

[12] M. Grassin.*Global smooth solutions to Euler equations for a perfect gas.* Indiana Univ. Math. J. 47 (1998), no. 4, 1397-1432.

[13] J. Ginibre, G.Velo, *Generalized Strichartz inequalities for the wave equation*, Funct. Anal. 133, (1995), no. 1, 50–68.

[14] L. Grafakos, *Classical and Modern Fourier Analysis*, Prentice Hall, New-York, 2006.

[15] T. Hmidi, *Low Mach number limit for the isentropic Euler system with axisymmetric initial data.* J. Inst. Math. Jussieu 12 (2013), no. 2, 335–389.

[16] T. Hmidi, S. Sulaiman, *Incompressible limit for the 2D isentropic Euler system with critical initial data*, preprint arXiv:

[17] S. Klainerman and A. Majda, *Compressible and incompressible fluids*, Comm. Pure Appl. Math., 35 (1982), no. 5, 629–651.

[18] S. Klainerman and A. Majda, *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids*, Comm. Pure Appl. Math., 34 (1981), no. 4, 481–524.

[19] C.-K. Lin. *On the incompressible limit of the slightly compressible viscous fluid flows. In Nonlinear waves* (Sapporo, 1995), volume 10 of GAKUTO Internat. Ser. Math. Sci. Appl., pages 277-282. Gakkotosho, Tokyo,1997.

[20] P.-L. Lions and N. Masmoudi. *Incompressible limit for a viscous compressible fluid.* J. Math. Pures Appl. (9), 77 (1998), no. 6, 585–627.

[21] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*, volume 53 of Applied Mathematical Sciences. Springer-Verlag, New York, 1984.

[22] G. Metivier and S. Schochet, *The incompressible limit of the non-isentropic Euler equations*, Arch. Ration. Mech. Anal., 158 (2001), no. 1, 61–90.

[23] A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics, vol. 27, Cambridge University Press, Cambridge, 2002.

[24] T. Makino, S. Ukai and S. Kawashima, *Sur la solution a support compact de l'equation d'Euler compressible* Japan J. Appl. Math. 3 (1986), no. 2, 249–257,

[25] E. Nakai and K. Yabuta, *Pointwise multipliers for functions of bounded mean oscillation*, J Math. Soc. Japan, 37 (2), 207–218 (1985).

[26] M. A. Rammaha, *Formation of singularities in compressible fluids in two-space dimensions.* Proc. Amer. Math. Soc. 107 (1989), no. 3, 705–714.

[27] D. Serre. *Solutions classiques globales des equations d'Euler pour un fluide parfait compressible.* Ann. Inst. Fourier(Grenoble) 47 (1997), no. 1, 139–153.

[28] C. T. Sideris, *Formation of singularities in three-dimensional compressible fluids.* Comm. Math. Phys. 101 (1985), no. 4, 475–485.

[29] C. T. Sideris, The lifespan of smooth solutions to the three-dimensional compressible Euler equations and the incompressible limit. Indiana Univ. Math. J. 40 (1991), no. 2, 535-550. 35Q30 (76N10)

[30] E. Stein, *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals.* Princeton University Press (1993).

[31] S. Ukai, *The incompressible limit and the initial layer of the compressible Euler equation.* J. Math. Kyoto Univ.,26 (1086), no. 2, 323–331.

[32] M. Vishik, *Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type.* (English, French summary) Ann. Sci. Ecole Norm. Sup. (4) 32 (1999), no. 6, 769-812.

[33] V.I. Yudovich, *Non-stationnary flows of an ideal incompressible fluid.* Zhurnal Vych Matematika, 3, pages 1032-106, 1963.

[34] V.I. Yudovich, *Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid.* Math. Res. Lett., 2 (1995), 27–38.

IRMAR, Université de Rennes 1, Campus de Beaulieu, 35 042 Rennes cedex, France
E-mail address: zineb.hassainia@univ-rennes1.fr